Study of equatorial ionospheric irregularities and mapping of electron density profiles and ionograms

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14. ABSTRACT
This report covers study of plasma bubbles through their spread echo in ionograms limiting to equatorial ionosphere, for the reasons: (1)part of the Philippines lies at the magnetic equator, (2) equatorial ionosphere is not well understood, (3) plasma bubbles form in the equatorial ionosphere, and (4) the equations are simpler because the magnetic field is nearly horizontal, so that the ionospheric layers are also nearly horizontal. Also study plasma bubbles to determine from an ionogram, the electron density as a function of true height which is not a very easy problem. Generate the ionogram as a plot of the virtual height of the ionosphere as a function of the carrier frequency of the radio wave. This study shall generate an atlas of all possible electron density profiles and derive the ionograms. Using experimentally obtained ionograms and comparing with the theoretical ionograms, the corresponding electron density profile of the ionogram shall be found.

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Chapter 1. Introduction

The project required a research team to investigate and study plasma bubbles through their spread echo in ionograms limiting to equatorial ionosphere, for the reasons: (1) part of the Philippines lies at the magnetic equator, (2) equatorial ionosphere is not well understood, (3) plasma bubbles form in the equatorial ionosphere, and (4) the equations are simpler because the magnetic field is nearly horizontal, so that the ionospheric layers are also nearly horizontal. Also study plasma bubbles to determine from an ionogram, the electron density as a function of true height which is not a very easy problem. Generate the ionogram as a plot of the virtual height of the ionosphere as a function of the carrier frequency of the radio wave. This study shall generate an atlas of all possible electron density profiles and derive the ionograms. Using experimentally obtained ionograms and comparing with the theoretical ionograms, the corresponding electron density profile of the ionogram shall be found.

This report consists of two papers that have been submitted for publication. They constitute Chapters 2 (Polarization ellipse and Stokes parameters in geometric algebra) and 3 (Refractive index and energy-momentum of elliptically polarized radio waves in plasma via geometric algebra).

A third paper will be submitted later:
Clint G. Bennett, Quirino M. Sugon Jr., and Daniel J. McNamara, "Simulation of electron density profiles and their ionograms for vertical incidence radio wave propagation in equatorial ionosphere."

Polarization ellipse and Stokes parameters in geometric algebra

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In this paper, we use geometric algebra to describe the polarization ellipse and Stokes parameters. We show that a solution to Maxwell's equation is a product of a complex basis vector in Jackson and a linear combination of plane wave functions. We convert both the amplitudes and the wave function arguments from complex scalars to complex vectors. This conversion allows us to separate the electric field vector and the imaginary magnetic field vector, because exponentials of imaginary scalars convert vectors to imaginary vectors and vice versa, while exponentials of imaginary vectors only rotate the vector or imaginary vector they are multiplied to. We convert this expression for polarized light into two other representations: the Cartesian representation and the rotated ellipse representation. We compute the conversion relations among the representation parameters and their corresponding Stokes parameters. And finally, we propose a set of geometric relations between the electric and magnetic fields that satisfy an equation similar to the Poincaré sphere equation.

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1. INTRODUCTION

There are many formalisms to describe polarized light: Stokes parameters [1,2], Jones calculus [3,4], Poincaré sphere [5], Mueller matrices [6], Pauli formalism [7–9], Jackson's complex vectors [10], complex numbers [11,12], and the more recent pure operatorial Pauli algebraic approach [13–15]. All these formalisms are dependent on each other, and there are fundamental textbooks [16,17] in polarization theory that establish the correspondences between them.

Many of these formalisms, however, use the imaginary number simply as a mathematical artifice for deriving equations: only the real part is physically meaningful; the imaginary part is discarded. Also, the magnetic field is not explicitly included in the description of polarized light. In this paper, we wish to show that imaginary numbers are also physically meaningful, especially in the description of electromagnetic waves and polarized light. We shall do this using geometric algebra.

Clifford (geometric) algebra was invented by Clifford in 1878 [18], but it was only applied to polarized light by Hestenes in 1971 [19], Baylis in 1993 [8], and Doran and Lasenby in 2003 [9]. Clifford algebra is compact and coherent. For example, the geometric product of two vectors may be expressed as a sum of their dot and cross products [20]:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + i\mathbf{a}\times\mathbf{b}, \tag{1}$$

where i is a geometric product of three orthonormal vectors. Except for the absence of the Pauli σ matrices, the geometric product in Eq. (1) is similar to the Pauli identity in quantum mechanics [21].

Using the identity in Eq. (1), we can define an electromagnetic field $\hat{E}$ as a sum of its electric field vector and its imaginary magnetic field vector:

$$\hat{E} = \mathbf{E} + i\zeta\mathbf{H}, \tag{2}$$

where $\zeta = \sqrt{\mu_0/\epsilon_0}$ is the ratio of $\mathbf{E}$ to $\mathbf{H}$ for the radiation field as given in Haus and Melcher [22]. This expression lets us define an elliptically polarized electromagnetic wave in terms of the exponential plane wave functions $e^{\pm i(\omega t - kz)}$:

$$\mathbf{E} + i\zeta\mathbf{H} = (\mathbf{e}_1 + i\mathbf{e}_2)[a_+ e^{-i(\omega t - kz + \delta_+)} + a_- e^{i(\omega t - kz - \delta_-)}]. \tag{3}$$

In Baylis [8], Hestenes [19], and in Doran and Lasenby [9], the electromagnetic field is given $\mathbf{F} = \mathbf{E} + i\mathbf{B}$, because the magnetic flux density $\mathbf{B}$ and not the magnetic field $\mathbf{H}$ is the physical magnetic field in the Lorentz force. But we are following the convention of Jancewicz [23] wherein the electromagnetic field is given by $\sqrt{\epsilon}\mathbf{E} + \sqrt{\mu}\mathbf{H}$. Dividing this by $\sqrt{\epsilon}$ would yield our expression in Eq. (2).

We shall not go into the Stokes subspace of the Poincaré sphere [8,24], which is spanned by the 2×2 Pauli spin matrices $\sigma_1 \equiv \underline{\mathbf{e}}_1$, $\sigma_2 \equiv \underline{\mathbf{e}}_2$, and $\sigma_3 \equiv \underline{\mathbf{e}}_3$. These matrices, which also satisfy the orthonormality relation of Clifford algebra, act on an abstract column vector defined as the spinor $(a_+ e^{-i\delta_+}, a_- e^{i\delta_-})$. Note that despite their formal similarity, the Pauli spin matrices are not equivalent to the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ that span the real physical space.

Doran and Lasenby [9] have defined polarized light as

$$\mathbf{E} = \mathrm{Re}((c_1\sigma_1 + c_2\sigma_2)e^{-ik\cdot x}) = \mathrm{Re}(\mathcal{E}). \tag{4}$$

We shall instead use the authors' exponential relations to transform the exponents in Eq. (2) from i to $-i\mathbf{e}_3$. This allows us to conveniently separate the vector and imaginary vector parts to yield the relations for the electric and magnetic fields:

$$\mathbf{E} = \mathbf{e}_1[a_+ e^{i\mathbf{e}_3(\omega t - kz + \delta_+)} + a_- e^{-i\mathbf{e}_3(\omega t - kz - \delta_-)}], \tag{5a}$$

$$\zeta\mathbf{H} = \mathbf{e}_2[a_+ e^{i\mathbf{e}_3(\omega t - kz + \delta_+)} + a_- e^{-i\mathbf{e}_3(\omega t - kz - \delta_-)}]. \tag{5b}$$

In this form, both the electric and magnetic fields are expressed as sums of counter-rotating vectors with radii a_+ and a_-, initial rotation angles δ_+ and δ_-, and rotation parameters t and z for time and space. That is, the exponential of an imaginary vector $e^{\pm i\mathbf{e}_3(\omega t - kz \pm \delta_\pm)}$ acts as a rotation operator to the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Using the rules of geometric algebra, we shall relate this form with two other known representations: (i) Cartesian representation whose component oscillations are along the x and y axes and (ii) rotated ellipse representation whose component oscillations are along the semimajor and semiminor axes of the polarization ellipse.

The Stokes parameters S_0, S_1, S_2, and S_3 are defined as four observable quantities that characterize the polarization state of light [25]. In the literature [3,26], there is no adequate summary of the Stokes parameters in the three representations of polarized light, so we shall present this summary here. We shall show that the Stokes parameters are related to the geometry of the polarization ellipse and that the first Stokes parameter S_0 corresponds to the magnitudes of the time-averaged energy and momentum densities. Furthermore, we shall define the field relation parameters:

$$s_0 = \mathbf{E}^2 + \zeta^2\mathbf{H}^2, \tag{6a}$$

$$s_1 = \mathbf{E}^2 - \zeta^2\mathbf{H}^2, \tag{6b}$$

$$is_2 = 2i\zeta\mathbf{E}\cdot\mathbf{H} = 2i\zeta|\mathbf{E}||\mathbf{H}|\cos\alpha, \tag{6c}$$

$$\mathbf{s}_3 = 2\zeta\mathbf{E}\times\mathbf{H} = 2\mathbf{e}_3\zeta|\mathbf{E}||\mathbf{H}|\sin\alpha, \tag{6d}$$

where α is the angle between the vectors $\mathbf{E}$ and $\mathbf{H}$. Although these parameters are not equivalent to the classical Stokes parameters, they are physically important because they define the relationships between the electric and magnetic fields that are not found in the classical Stokes parameters. Moreover, the field relation parameters satisfy an equation similar to that of the Poincaré sphere:

$$s_0^2 = s_1^2 + s_2^2 + \mathbf{s}_3^2. \tag{7}$$

This paper is divided into six sections. Section 1 is the introduction. In Section 2, we shall discuss geometric products of vectors and rotation operators in geometric algebra $\mathcal{C}l_{3,0}$. These operators are expressed in terms of the exponentials of imaginary numbers and of imaginary vectors. In Section 3, we discuss Maxwell's equation and its electromagnetic wave solutions expressed as a sum of products of the circularly polarized basis and wave functions. We express in geometric algebra terms three classical representations of the polarization ellipse: Cartesian [16], circular [17], and rotated ellipse representations. In Section 4, we shall present the classical Stokes parameters in all three representations. We shall also propose a set of relations between the electric and magnetic fields in terms of four parameters: the sum and difference of their magnitudes and their vector dot and cross products. We shall show that these magnitudes are related to the electromagnetic energy and momentum densities and the orthogonality and magnitude relations between the electric and magnetic fields. Section 5 is the conclusion.

2. GEOMETRIC ALGEBRA

In this section we shall introduce the products of basis vectors and complex numbers in geometric algebra and use these to derive the associative geometric product of vectors as a sum of their dot and imaginary cross products. By doing so, we shall show that a general form of an element in geometric algebra for three-dimensional space consists of a scalar, vector, imaginary vector, and an imaginary scalar, which corresponds to a magnitude, an oriented line segment, an oriented plane, and an oriented volume, respectively. After this, we shall develop the formalism for vector rotations. We shall start with exponentials of imaginary scalars as rotation operators of complex vectors. Then we shall proceed with exponentials of imaginary vectors as rotation operators of vectors.

A. Vectors and Imaginary Numbers

There are many vector spaces in which geometric algebras can be generated, but we shall focus on the geometric algebra $\mathcal{C}l_{3,0}$, also known as the Pauli algebra [23], which is generated by the three orthonormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ of the physical space. Any product of these three vectors satisfy the following orthonormality axiom:

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = 2\delta_{jk}. \tag{8}$$

This means that

$$\mathbf{e}_j^2 = \mathbf{e}_k^2 = 1, \tag{9a}$$

$$\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j, \qquad j \neq k. \tag{9b}$$

In other words, parallel vectors commute while perpendicular vectors anticommute.

The algebra also defines a trivector i,

$$i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \tag{10}$$

which has the following properties:

$$i^2 = -1, \tag{11}$$

$$i\mathbf{e}_1 = \mathbf{e}_1 i = \mathbf{e}_2\mathbf{e}_3, \tag{12a}$$

$$i\mathbf{e}_2 = \mathbf{e}_2 i = \mathbf{e}_3\mathbf{e}_1, \tag{12b}$$

$$i\mathbf{e}_3 = \mathbf{e}_3 i = \mathbf{e}_1\mathbf{e}_2. \tag{12c}$$

Equation (11) states that the trivector i is an imaginary number. Equations (12a)–(12c), on the other hand, state that i not only commutes with vectors but also transforms these vectors into oriented planes perpendicular to them. These planes are called bivectors.

B. Vector Products and Multivectors

Let vectors $\mathbf{a}$ and $\mathbf{b}$ be defined as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \tag{13a}$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3. \tag{13b}$$

Their product is

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}, \tag{14}$$

where

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3, \tag{15a}$$

$$\mathbf{a}\wedge\mathbf{b} = \mathbf{e}_1\mathbf{e}_2(a_1b_2 - a_2b_1) + \mathbf{e}_2\mathbf{e}_3(a_2b_3 - a_3b_2) + \mathbf{e}_3\mathbf{e}_1(a_3b_1 - a_1b_3). \tag{15b}$$

Notice that the product of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is the sum of a scalar dot product $\mathbf{a}\cdot\mathbf{b}$ and a bivector wedge product $\mathbf{a}\wedge\mathbf{b}$. Note that $\mathbf{a}\cdot\mathbf{b}$ is symmetric and $\mathbf{a}\wedge\mathbf{b}$ is antisymmetric.

To relate the wedge product $\mathbf{a}\wedge\mathbf{b}$ to the more familiar cross product $\mathbf{a}\times\mathbf{b}$, we use the relations in Eqs. (12a)–(12c) to obtain

$$\mathbf{a}\wedge\mathbf{b} = i\mathbf{a}\times\mathbf{b}. \tag{16}$$

Here, the bivector (oriented plane) $\mathbf{a}\wedge\mathbf{b}$ is rewritten in terms of the pseudovector $\mathbf{a}\times\mathbf{b}$ normal to the plane formed by $\mathbf{a}$ and $\mathbf{b}$.

Geometric algebra allows elements belonging to different vector spaces to be added up; e.g., the geometric product is the sum of a scalar and an imaginary vector. This means that the most general form of the sum can be constructed by having a linear combination of scalars, vectors, imaginary vectors, and imaginary scalars:

$$\hat{A} = a_0 + \mathbf{a}_1 + i\mathbf{a}_2 + ia_3, \tag{17}$$

where a_0 and a_3 are scalars and $\mathbf{a}_1$ and $\mathbf{a}_2$ are vectors. The spatial inverse $\hat{A}^\dagger$ of $\hat{A}$ is obtained by reversing the sign of every vector in $\hat{A}$:

$$\hat{A}^\dagger = a_0 - \mathbf{a}_1 + i\mathbf{a}_2 - ia_3, \tag{18}$$

which is similar to the parity operation in Sakurai [27] except that we define $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, which changes sign under the parity operation. Notice that scalars and imaginary vectors do not change sign under the parity operation, while vectors and imaginary scalars do. Note that the dagger notation used here is not the same as the reversion operator, such as in Hestenes and Sobczyk [20], which reverses the order of the vector factors, so that the reverse of $\hat{A}$ is $a_0 - \mathbf{a}_1 - i\mathbf{a}_2 - ia_3$. This is not the same as Eq. (18). The definition of spatial inversion in Eq. (18) is actually the same as the automorphic grade involution in Baylis [8].

C. Vector Rotation via Exponentials of Trivectors

Because $i^2 = -1$, then by Euler's theorem,

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta, \tag{19}$$

which is a sum of a scalar and a trivector. From complex analysis, we know that

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \tag{20a}$$

$$\sin\theta = -\frac{i}{2}(e^{i\theta} - e^{-i\theta}). \tag{20b}$$

To rotate vectors using the exponential $e^{\pm i\theta}$, we cannot use ordinary vectors as operands. Instead, we use the complex vector in Jackson [10]:

$$\mathbf{e}_+ = \mathbf{e}_1 + i\mathbf{e}_2, \tag{21}$$

and we extract the real part later. That is, if we define

$$\mathbf{e}_{1\mp} + i\mathbf{e}_{2\pm} = (\mathbf{e}_1 + i\mathbf{e}_2)e^{\pm i\theta}, \tag{22}$$

and we use Eq. (19) in Eq. (22), we obtain

$$\mathbf{e}_{1\mp} = \mathbf{e}_1\cos\theta \mp \mathbf{e}_2\sin\theta, \tag{23a}$$

$$\mathbf{e}_{2\pm} = \pm\mathbf{e}_1\sin\theta + \mathbf{e}_2\cos\theta, \tag{23b}$$

after separating the scalar and bivector parts. Geometrically, $\mathbf{e}_{1+}$ is a rotation of $\mathbf{e}_1$ about $\mathbf{e}_3$ counterclockwise by an angle θ, while $\mathbf{e}_{1-}$ is similar except that the rotation is clockwise. On the other hand, $\mathbf{e}_{2+}$ is a rotation of $\mathbf{e}_2$ about $\mathbf{e}_3$ clockwise by an angle θ, while $\mathbf{e}_{2-}$ is similar except that the rotation is counterclockwise (Fig. 1).

D. Vector Rotation via Exponentials of Bivectors

Because $(i\mathbf{e}_3)^2 = (\mathbf{e}_1\mathbf{e}_2)^2 = -1$, then by Euler's theorem [28],

$$e^{\pm i\mathbf{e}_3\theta} = \cos\theta \pm i\mathbf{e}_3\sin\theta = \cos\theta \pm \mathbf{e}_1\mathbf{e}_2\sin\theta, \tag{24}$$

which is a sum of a scalar and a bivector. From complex analysis, the relations corresponding to Eqs. (20a) and (20b) are

$$\cos\theta = \frac{1}{2}(e^{i\mathbf{e}_3\theta} + e^{-i\mathbf{e}_3\theta}), \tag{25a}$$

$$\sin\theta = -\frac{1}{2}i\mathbf{e}_3(e^{i\mathbf{e}_3\theta} - e^{-i\mathbf{e}_3\theta}). \tag{25b}$$

The product of $\mathbf{e}_1$ and $\mathbf{e}_2$ with the bivector $i\mathbf{e}_3$ is also a rotation of these two vectors about $\mathbf{e}_3$ counterclockwise by an angle $\pi/2$:

$$\mathbf{e}_1 i\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2, \tag{26a}$$

$$\mathbf{e}_2 i\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1, \tag{26b}$$

where we used the relation in Eq. (12c). In general, if we want to rotate $\mathbf{e}_1$ and $\mathbf{e}_2$ about $\mathbf{e}_3$ arbitrarily by an angle θ, we right-multiply these two vectors by $e^{\pm i\mathbf{e}_3\theta}$ in Eq. (24), where the negative sign means clockwise rotation and the positive sign means counterclockwise rotation:

$$\mathbf{e}_1 e^{\pm i\mathbf{e}_3\theta} = \mathbf{e}_1 \cos\theta \pm \mathbf{e}_2 \sin\theta, \tag{27a}$$

$$\mathbf{e}_2 e^{\pm i\mathbf{e}_3\theta} = \mp\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta. \tag{27b}$$

Comparing Eqs. (27a), (27b), (23a), and (23b), we get

$$\mathbf{e}_{1\mp} = \mathbf{e}_1 e^{\mp i\mathbf{e}_3\theta}, \tag{28a}$$

$$\mathbf{e}_{2\pm} = \mathbf{e}_2 e^{\mp i\mathbf{e}_3\theta}. \tag{28b}$$

Adding Eqs. (28a) and (28b), and using Eq. (22), we arrive at

$$(\mathbf{e}_1 + i\mathbf{e}_2)e^{\pm i\theta} = (\mathbf{e}_1 + i\mathbf{e}_2)e^{\mp i\mathbf{e}_3\theta}. \tag{29}$$

Equation (29) is the exponential conversion relations for vector rotations.

The products of the unit vectors with the exponential $e^{\pm i\mathbf{e}_3\theta}$ are

$$\mathbf{e}_1 e^{\pm i\mathbf{e}_3\theta} = e^{\mp i\mathbf{e}_3\theta}\mathbf{e}_1, \tag{30a}$$

$$\mathbf{e}_2 e^{\pm i\mathbf{e}_3\theta} = e^{\mp i\mathbf{e}_3\theta}\mathbf{e}_2, \tag{30b}$$

$$\mathbf{e}_3 e^{\pm i\mathbf{e}_3\theta} = e^{\pm i\mathbf{e}_3\theta}\mathbf{e}_3. \tag{30c}$$

Notice that $\mathbf{e}_1$ and $\mathbf{e}_2$ change the sign of the exponent upon swapping while $\mathbf{e}_3$ will not change the sign of the exponent.

3. ELECTROMAGNETIC THEORY

In this section, we show how the four Maxwell's equations in free space may be combined in a single equation that states that the space–time derivative of the electromagnetic field is zero, where the vector electromagnetic field is a sum of a vector electric field and a bivector magnetic field. We shall show that a solution to Maxwell's equation is a circularly polarized electromagnetic wave, and from this we construct an elliptically polarized wave solution. Using geometric algebra, we shall transform the elliptically polarized solution from an expression containing exponentials of trivectors to another containing exponentials of bivectors. After this, we convert the circular basis representation of elliptically polarized light to the Cartesian and rotated ellipse representations.

A. Maxwell's Equation

Maxwell's equation for electromagnetic fields in free space can be expressed in geometric algebra as a single equation [29,30]:

$$\frac{\partial \hat{E}}{\partial \hat{r}} = 0, \tag{31}$$

where

$$\hat{E} = \mathbf{E} + i\zeta\mathbf{H}, \tag{32a}$$

$$\frac{\partial}{\partial \hat{r}} = \frac{1}{c}\frac{\partial}{\partial t} + \nabla \tag{32b}$$

are the electromagnetic field [29,31] and the space–time derivative operator, respectively. Distributing the space–time derivative operator and applying the definition of the geometric product to the nabla operator, we get

$$\nabla \cdot \mathbf{E} = 0, \tag{33a}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \zeta\nabla \times \mathbf{H} = 0, \tag{33b}$$

$$\nabla \times \mathbf{E} + \frac{\zeta}{c}\frac{\partial \mathbf{H}}{\partial t} = 0, \tag{33c}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{33d}$$

after separating the scalar, vector, bivector, and trivector parts. Equations (33a)–(33d) are Gauss's law, Ampere's law, Faraday's Law, and the magnetic flux continuity law, respectively.

B. Circularly and Elliptically Polarized Electromagnetic Waves

We assume the following electromagnetic plane wave solution to Maxwell's equation:

$$\hat{E}_\pm = \mathbf{E}_\pm + i\zeta\mathbf{H}_\pm = \mathbf{e}_+ \tilde{a}_\pm \tilde{\psi}^{\pm 1}, \tag{34}$$

where

$$\mathbf{e}_+ = \mathbf{e}_1 + i\mathbf{e}_2, \tag{35a}$$

$$\tilde{a}_\pm = a_\pm e^{i\delta_\pm}, \tag{35b}$$

$$\tilde{\psi}^{\pm 1} = e^{\mp i(\omega t - kz)}. \tag{35c}$$

Here, $\tilde{a}_\pm$ is the complex amplitude and $\tilde{\psi}^{\pm 1}$ is the wave function. Note that though we use a complex vector $\mathbf{e}_+$ that is proportional to Jackson's $\epsilon_+ = \mathbf{e}_+/\sqrt{2}$, we shall not use his other basis $\epsilon_- = (\mathbf{e}_1 - i\mathbf{e}_2)/\sqrt{2}$; the complex vector $\mathbf{e}_+$ is sufficient for our purposes.

In order to understand the physical meaning of Eq. (34), we substitute Eqs. (35a)–(35c) into our solution,

$$\hat{E}_\pm = \mathbf{E}_\pm + i\zeta\mathbf{H}_\pm = (\mathbf{e}_1 + i\mathbf{e}_2)a_\pm e^{\mp i(\omega t - kz \mp \delta_\pm)}. \tag{36}$$

Separating the vector and bivector parts, we arrive at

$$\mathbf{E}_\pm = \mathbf{e}_1 a_\pm \cos(\omega t - kz \mp \delta_\pm) \pm \mathbf{e}_2 a_\pm \sin(\omega t - kz \mp \delta_\pm), \tag{37a}$$

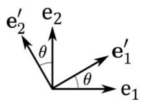

Fig. 1. Rotation of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ counterclockwise by an angle θ about the unit vector $\mathbf{e}_3$. The new vectors are $\mathbf{e}_1' = \mathbf{e}_1 e^{i\mathbf{e}_3\theta}$ and $\mathbf{e}_2' = \mathbf{e}_2 e^{i\mathbf{e}_3\theta}$. The unit vector $\mathbf{e}_3$ is pointing out of the page.

$$\zeta \mathbf{H}_\pm = \mathbf{e}_2 a_\pm \cos(\omega t - kz \mp \delta_\pm) \mp \mathbf{e}_1 a_\pm \sin(\omega t - kz \mp \delta_\pm). \quad (37b)$$

Equations (37a) and (37b) describe circularly polarized electric and magnetic fields. If the subscript of $\mathbf{E}$ and $\mathbf{H}$ is positive, then the polarization is left-circularly polarized. If negative, the polarization is right-circularly polarized.

To make the expressions for these fields more compact, we use the theorem in Eq. (29) for Eq. (36) to get

$$\begin{aligned}\hat{E}_\pm &= \mathbf{E}_\pm + i\zeta \mathbf{H}_\pm = (\mathbf{e}_1 + i\mathbf{e}_2) a_\pm e^{\pm i\mathbf{e}_3(\omega t - kz \pm \delta_\pm)} \\ &= \mathbf{e}_1 a_\pm e^{\pm i\mathbf{e}_3(\omega t - kz \pm \delta_\pm)} \pm i\mathbf{e}_2 a_\pm e^{\pm i\mathbf{e}_3(\omega t - kz \pm \delta_\pm)}. \end{aligned} \quad (38)$$

Separating the vector and bivector parts, we obtain

$$\mathbf{E}_\pm = \mathbf{e}_1 a_\pm e^{\pm i\mathbf{e}_3(\omega t - kz \pm \delta_\pm)}, \quad (39a)$$

$$\zeta \mathbf{H}_\pm = \mathbf{e}_2 a_\pm e^{\pm i\mathbf{e}_3(\omega t - kz \pm \delta_\pm)}, \quad (39b)$$

after factoring out i in the second equation. Notice that the electric field $\mathbf{E}_\pm$, the magnetic field $\mathbf{H}_\pm$, and the wave vector $\mathbf{k} = k\mathbf{e}_3$ are mutually perpendicular, and that $\mathbf{E}_\pm$ and $\mathbf{H}_\pm$ are rotating about $\mathbf{k}$ in time t for a fixed position $\mathbf{r} = r_0 \mathbf{e}_3$ (Fig. 2). If the sign of the exponential argument before $i\mathbf{e}_3$ is positive, then the rotation is counterclockwise in time; if negative, then the rotation is clockwise.

Normally in the literature, instead of $\mathbf{e}_+ = \mathbf{e}_1 + i\mathbf{e}_2$, only $\mathbf{e}_1$ is used, so that instead of the complex monochromatic electromagnetic plane wave in Eq. (36), we have

$$\tilde{\mathbf{E}}_\pm = \mathbf{e}_1 a_\pm e^{\mp i(\omega t - kz \mp \delta_\pm)}. \quad (40)$$

Separating the vector and imaginary vector parts, we get

$$\mathrm{Re}(\tilde{\mathbf{E}}_\pm) = \mathbf{e}_1 a_\pm \cos(\omega t - kz \mp \delta_\pm), \quad (41a)$$

$$\mathrm{Im}(\tilde{\mathbf{E}}_\pm) = \mp i\mathbf{e}_1 a_\pm \sin(\omega t - kz \mp \delta_\pm). \quad (41b)$$

By convention, only the real part describes the electric field and the imaginary part is discarded, so that Eq. (40) only describes a linearly polarized light. Comparing this result with the circularly polarized wave expression in Eq. (36), we see the conciseness and completeness of geometric algebra that

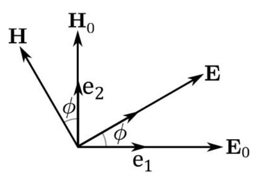

Fig. 2. Rotation of the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ with zero initial phase angle. The initial field vectors $\mathbf{E}_0$ and $\mathbf{H}_0$ point along the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions and are rotated by an angle $\phi = \omega t - kz$. The wave vector $\mathbf{k}$ is pointing out of the page.

uses the imaginary number i to unify the electric and magnetic fields and convert one to the other as the wave propagates in space and time as shown by Eqs. (37a) and (37b). Thus, in geometric algebra, a circularly polarized light is a more natural expression for the polarized monochromatic electromagnetic plane wave.

Now that we have established that Eq. (34) is a circularly polarized electromagnetic wave, we now verify if it does indeed satisfies Maxwell's equation in Eq. (31). Substituting Eq. (34) to Maxwell's equation in Eq. (31), we get

$$\frac{\partial \hat{E}_\pm}{\partial \hat{r}} = \mp i\left(\frac{\omega}{c} - k\mathbf{e}_3\right)(\mathbf{e}_1 + i\mathbf{e}_2)\tilde{a}_\pm \tilde{\psi}^{\pm 1} = 0. \quad (42)$$

Because $\omega/c = k$ by the wave condition [29] and

$$(1 - \mathbf{e}_3)(\mathbf{e}_1 + i\mathbf{e}_2) = (1 - \mathbf{e}_3)(1 + \mathbf{e}_3)\mathbf{e}_1 = 0, \quad (43)$$

then the electromagnetic wave Eq. (34) is indeed a solution to Maxwell's equation in Eq. (31).

We may also rewrite Eq. (42) in a coordinate-free way as

$$\frac{\partial \hat{E}_\pm}{\partial \hat{r}} = \mp i\left(\frac{\omega}{c} - \mathbf{k}\right)(\mathbf{E}_\pm + i\zeta \mathbf{H}_\pm) = 0, \quad (44)$$

where we used the relations in Eq. (34) together with $\mathbf{k} = k\mathbf{e}_3$. Factoring out the trivector i in Eq. (44) and separating the scalar, vector, bivector, and trivector parts, we get [30]

$$\mathbf{k} \cdot \mathbf{E}_\pm = 0, \quad (45a)$$

$$\frac{\omega}{c}\mathbf{E}_\pm + \zeta \mathbf{k} \times \mathbf{H}_\pm = 0, \quad (45b)$$

$$i\frac{\omega}{c}\zeta \mathbf{H}_\pm - i\mathbf{k} \times \mathbf{E}_\pm = 0, \quad (45c)$$

$$i\zeta \mathbf{k} \cdot \mathbf{H}_\pm = 0. \quad (45d)$$

Because the electric field $\mathbf{E}_\pm$ and the magnetic field $\mathbf{H}_\pm$ are both perpendicular to the wave vector $\mathbf{k}$ as given in Eqs. (45a) and (45d), then we may rewrite the cross products of $\mathbf{k}$ with $\mathbf{E}_\pm$ and $\mathbf{H}_\pm$ in terms of the geometric product:

$$\mathbf{E}_\pm - i\zeta \frac{\mathbf{k}}{|\mathbf{k}|}\mathbf{H}_\pm = 0, \quad (46a)$$

$$\zeta \mathbf{H}_\pm + i\frac{\mathbf{k}}{|\mathbf{k}|}\mathbf{E}_\pm = 0, \quad (46b)$$

after factoring out $\omega/c = |\mathbf{k}|$. Notice that these two equations are equal. For example, if we left multiply Eq. (46b) by $-i\mathbf{k}/|\mathbf{k}|$ we get Eq. (46a), because $\mathbf{k}^2 = |\mathbf{k}|^2$.

Using Eq. (46b), we may rewrite the electromagnetic field $\hat{E}_\pm$ as

$$\hat{E}_\pm = \mathbf{E}_\pm + i\zeta \mathbf{H}_\pm = \left(1 + \frac{\mathbf{k}}{|\mathbf{k}|}\right)\mathbf{E}_\pm. \quad (47)$$

From this we see that $i\zeta\mathbf{H}_\pm$ is the same as the bivector or oriented plane formed by the unit vector $\mathbf{k}/|\mathbf{k}|$ and electric field $\mathbf{E}_\pm$:

$$i\zeta\mathbf{H}_\pm = \frac{\mathbf{k}}{|\mathbf{k}|}\mathbf{E}_\pm = \frac{\mathbf{k}}{|\mathbf{k}|}\wedge\mathbf{E}_\pm. \quad (48)$$

C. Representations of Polarization Ellipse

In general, the solution to Maxwell's equation with frequency ω is a linear superposition of wave functions $\tilde{\psi}^{-1}$ and $\tilde{\psi}$ [29]:

$$\hat{E} = \hat{\mathbf{e}}_+(\tilde{a}_+\tilde{\psi} + \tilde{a}_-\tilde{\psi}^{-1}), \quad (49)$$

which can be shown to be an expression for elliptically polarized light, if we only work in the complex plane by plotting only the curve traced by the expression $\tilde{a}_+\tilde{\psi} + \tilde{a}_-\tilde{\psi}^{-1}$ as discussed in Azzam and Bashara [16] and Klein and Furtak [12]. We use Eq. (29) to transform the general electromagnetic wave expression in Eq. (49) to obtain

$$\hat{E} = \hat{\mathbf{e}}_+(\hat{a}_+\hat{\psi} + \hat{a}_-\hat{\psi}^{-1}), \quad (50)$$

where

$$\hat{a}_\pm = a_\pm e^{i\mathbf{e}_3\delta_\pm}, \quad (51a)$$

$$\hat{\psi}^{\pm 1} = e^{\pm i\mathbf{e}_3(\omega t - kz)}. \quad (51b)$$

Note that $\tilde{\psi} = e^{-i(\omega t - kz)}$ in Eq. (49) is an exponential of an imaginary number or trivector, while $\hat{\psi} = e^{i\mathbf{e}_3(\omega t - kz)}$ in Eq. (50) is an exponential of an imaginary vector or bivector.

Separating the vector and imaginary vector parts of the transformed electromagnetic wave expression in Eq. (50) yields the electric and magnetic fields:

$$\mathbf{E} = \mathbf{e}_1(\hat{a}_+\hat{\psi} + \hat{a}_-\hat{\psi}^{-1}), \quad (52a)$$

$$\zeta\mathbf{H} = \mathbf{e}_2(\hat{a}_+\hat{\psi} + \hat{a}_-\hat{\psi}^{-1}). \quad (52b)$$

Because the electric and magnetic fields have the same elliptically polarized wave expression except that the unit vectors rotated for the electric field is $\mathbf{e}_1$ and that for the magnetic field is $\mathbf{e}_2$, then it is sufficient, from this point onward, to discuss only the electric field.

Using the definitions of $\hat{a}_\pm$ and $\hat{\psi}^{\pm 1}$ in Eqs. (51a) and (51b), we may rewrite the electric field in Eq. (52a) as

$$\mathbf{E} = \mathbf{e}_1[a_+ e^{i\mathbf{e}_3(\phi+\delta_+)} + a_- e^{-i\mathbf{e}_3(\phi-\delta_-)}], \quad (53)$$

where

$$\phi = \omega t - kz. \quad (54)$$

Equation (53) is the circular basis representation of elliptically polarized light, interpreted as a sum of two vectors of lengths a_+ and a_- that are rotating counterclockwise and clockwise, respectively, with initial phase angles δ_+ and δ_- from the positive x axis (Fig. 3). Notice that the exponentials $e^{i\mathbf{e}_3(\phi+\delta_+)}$ and $e^{-i\mathbf{e}_3(\phi-\delta_-)}$ act as vector rotation operators on the vectors $a_+\mathbf{e}_1$ and $a_-\mathbf{e}_1$. Thus, unlike in Klein and Furtak [12], which uses phasors in complex analysis as an accessory to vector analysis, we do not take the real vector part and discard the imaginary vector part of the electric field $\mathbf{E}$ because the result of the rotations is still a vector.

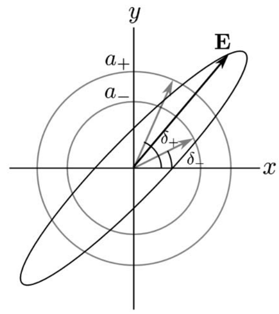

Fig. 3. Circular basis representation of the elliptically polarized light $\mathbf{E}$ as the sum of two vectors rotating in opposite directions with radii a_+ and a_- and initial phase angles δ_+ and δ_-. The semimajor axis is $E_1 = a_+ + a_-$, the semiminor axis is $E_2 = a_+ - a_-$, and the tilt angle is $\beta/2 = (\delta_+ + \delta_-)/2$.

Using Euler's theorem in Eq. (24) and the identity $\mathbf{e}_1 i\mathbf{e}_3 = \mathbf{e}_2$ in Eq. (26a), Eq. (53) becomes

$$\begin{aligned}\mathbf{E} = {} & \mathbf{e}_1(a_+\cos(\phi+\delta_+) + a_-\cos(\phi-\delta_-)) \\ & + \mathbf{e}_2(a_+\sin(\phi+\delta_+) - a_-\sin(\phi-\delta_-)).\end{aligned} \quad (55)$$

This may be rewritten as

$$\mathbf{E} = E_x\cos(\phi+\delta_x)\mathbf{e}_1 + E_y\cos(\phi+\delta_y)\mathbf{e}_2, \quad (56)$$

where

$$E_x = [a_+^2 + a_-^2 + 4a_+a_-\cos(\delta_+ + \delta_-)]^{\frac{1}{2}}, \quad (57a)$$

$$E_y = [a_+^2 + a_-^2 - 4a_+a_-\cos(\delta_+ + \delta_-)]^{\frac{1}{2}}, \quad (57b)$$

$$\delta_x = \arctan\left(\frac{a_+\sin\delta_+ - a_-\sin\delta_-}{a_+\cos\delta_+ + a_-\cos\delta_-}\right), \quad (57c)$$

$$\delta_y = \arctan\left(\frac{a_-\cos\delta_- - a_+\cos\delta_+}{a_+\sin\delta_+ + a_-\sin\delta_-}\right). \quad (57d)$$

Equation (56) is the Cartesian basis representation of elliptically polarized light, interpreted as a sum of two orthogonal oscillations with E_x and E_y as the amplitudes along the x and y axes, respectively, with δ_x and δ_y as the initial phase angles of oscillation (Fig. 4). Equations (64a) and (57d) are the expressions of the Cartesian basis parameters in terms of the circular basis parameters. Their inverse relations are

$$a_\pm = \frac{1}{2}[E_x^2 \mp 2E_xE_y\sin(\pm(\delta_x - \delta_y)) + E_y^2]^{\frac{1}{2}}, \quad (58a)$$

$$\delta_+ = \arctan\left(\frac{E_x\cos\delta_x - E_y\cos\delta_y}{E_x\sin\delta_x + E_y\sin\delta_y},\right), \quad (58b)$$

$$\delta_- = -\arctan\left(\frac{E_x\sin\delta_x - E_y\cos\delta_y}{E_x\cos\delta_x + E_y\sin\delta_y}\right). \quad (58c)$$

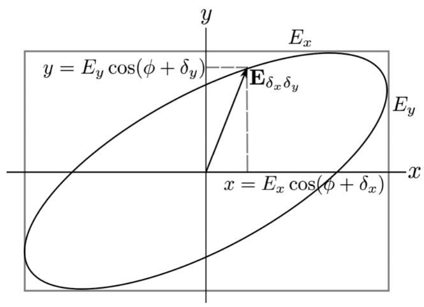

Fig. 4. Cartesian basis representation of elliptically polarized light $\mathbf{E}_{\delta_x\delta_y}$ as a sum of two orthogonal oscillations with amplitudes E_x and E_y, and initial phase angles δ_x and δ_y.

On the other hand, if we define

$$a_\pm = \frac{1}{2}(E_1 \pm E_2), \tag{59a}$$

$$\delta_\pm = \pm\delta + \frac{\beta}{2}, \tag{59b}$$

and substitute these relations into Eq. (53), we get

$$\mathbf{E} = \mathbf{e}_1\left[\frac{1}{2}(E_1+E_2)e^{i\mathbf{e}_3(\phi+\delta)} + \frac{1}{2}(E_1-E_2)e^{-i\mathbf{e}_3(\phi+\delta)}\right]e^{i\mathbf{e}_3\beta/2}. \tag{60}$$

Using Euler's theorem in Eq. (24) and the identity $\mathbf{e}_1 i\mathbf{e}_3 = \mathbf{e}_2$ in Eqs. (26a) and (60) becomes

$$\mathbf{E} = [E_1\cos(\phi+\delta)\mathbf{e}_1 + E_2\sin(\phi+\delta)\mathbf{e}_2]e^{i\mathbf{e}_3\beta/2}. \tag{61}$$

Equation (61) is the rotated ellipse representation of elliptically polarized light, interpreted as an ellipse with initial phase angle δ, semimajor axes lengths E_1 and E_2, that is rotated counterclockwise about $\mathbf{e}_3$ by an angle $\beta/2$ (Fig. 5). The advantage of using geometric algebra in this representation is that the ellipse $E_1\cos(\phi+\delta)\mathbf{e}_1 + E_2\sin(\phi+\delta)\mathbf{e}_2$ can be easily rotated via the action of the exponential of the bivector $i\mathbf{e}_3$. Equations (59a) and (59b) are the circular basis parameters in terms of the rotated ellipse representation parameters. The inverse relations are

$$E_1 = a_+ + a_-, \tag{62a}$$

$$E_2 = a_+ - a_-, \tag{62b}$$

$$\delta = \frac{1}{2}(\delta_+ - \delta_-), \tag{62c}$$

$$\frac{\beta}{2} = \frac{1}{2}(\delta_+ + \delta_-). \tag{62d}$$

For completeness, we express the rotated ellipse parameters (E_1, E_2, δ, $\beta/2$) in terms of the Cartesian basis parameters (E_x, E_y, δ_x, δ_y):

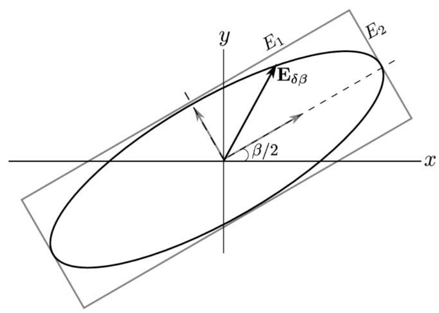

Fig. 5. Rotated ellipse representation of the elliptically polarized light $\mathbf{E}_{\delta\beta}$ as a rotated ellipse with semimajor axis E_1, semiminor axis E_2 and tilt angle $\beta/2$.

$$E_1 = \frac{1}{2}\Big[(E_x^2 + 2E_xE_y\sin(\delta_x-\delta_y) + E_y^2)^{\frac{1}{2}} + (E_x^2 - 2E_xE_y\sin(\delta_x-\delta_y) + E_y^2)^{\frac{1}{2}}\Big], \tag{63a}$$

$$E_2 = \frac{1}{2}\Big[(E_x^2 + 2E_xE_y\sin(\delta_x-\delta_y) + E_y^2)^{\frac{1}{2}} - (E_x^2 - 2E_xE_y\sin(\delta_x-\delta_y) + E_y^2)^{\frac{1}{2}}\Big], \tag{63b}$$

$$\delta = \frac{1}{2}\left[\arctan\left(\frac{E_x\cos\delta_x - E_y\cos\delta_y}{E_x\sin\delta_x + E_y\sin\delta_y}\right) + \arctan\left(\frac{E_x\sin\delta_x - E_y\cos\delta_y}{E_x\cos\delta_x + E_y\sin\delta_y}\right)\right], \tag{63c}$$

$$\frac{\beta}{2} = \frac{1}{2}\left[\arctan\left(\frac{E_x\cos\delta_x - E_y\cos\delta_y}{E_x\sin\delta_x + E_y\sin\delta_y}\right) - \arctan\left(\frac{E_x\sin\delta_x - E_y\cos\delta_y}{E_x\cos\delta_x + E_y\sin\delta_y}\right)\right]. \tag{63d}$$

Their inverse relations are

$$E_x = \left[\frac{E_1^2 + E_2^2 + (E_1^2 - E_2^2)\cos\beta}{2}\right]^{\frac{1}{2}}, \tag{64a}$$

$$E_y = \left[\frac{E_1^2 + E_2^2 - (E_1^2 - E_2^2)\cos\beta}{2}\right]^{\frac{1}{2}}, \tag{64b}$$

$$\delta_x = \arctan\left(\frac{E_1\sin\delta\cos\beta/2 + E_2\cos\delta\sin\beta/2}{E_1\cos\delta\cos\beta/2 - E_2\sin\delta\sin\beta/2}\right), \tag{64c}$$

$$\delta_y = \arctan\left(\frac{E_1\sin\delta\sin\beta/2 - E_2\cos\delta\cos\beta/2}{E_1\cos\delta\sin\beta/2 + E_2\sin\delta\cos\beta/2}\right). \tag{64d}$$

4. STOKES AND FIELD RELATION PARAMETERS

In this section, we derive the Stokes parameters in the rotated ellipse and circular basis representations starting with the Stokes parameters in the Cartesian basis representation, by using the basis parameter relations that are given in the previous section. After this we discuss the field relation parameters that also satisfy an equation similar to the Poincaré sphere equation.

A. Classical Stokes Parameters

The Stokes parameters in the Cartesian basis representation in Eq. (56) are widely known in the literature. They are defined as [11,32,33]

$$S_0 = E_x^2 + E_y^2, \tag{65a}$$

$$S_1 = E_x^2 - E_y^2, \tag{65b}$$

$$S_2 = 2E_xE_y\cos(\delta_x - \delta_y), \tag{65c}$$

$$S_3 = 2E_xE_y\sin(\delta_x - \delta_y). \tag{65d}$$

The parameter S_0 describes the total intensity of the optical field, S_1 describes the preponderance of linearly horizontally polarized light over linearly vertically polarized light, S_2 describes the preponderance of linear $+45°$ polarized light over linear $-45°$ polarized light, and the fourth parameter S_3 describes the preponderance of right-circularly polarized light over left-circularly polarized light [25]. The definition of the Stokes parameters in Eqs. (65a)–(65d) will be used in deriving the Stokes parameters for the rotated ellipse and the circular basis representations.

We can show that the Stokes parameters in Eqs. (65a)–(65d) can be expressed in the rotated ellipse representation $(E_1, E_2, \delta, \beta/2)$ as

$$S_0 = E_1^2 + E_2^2, \tag{66a}$$

$$S_1 = (E_1^2 - E_2^2)\cos\beta, \tag{66b}$$

$$S_2 = (E_1^2 - E_2^2)\sin\beta, \tag{66c}$$

$$S_3 = 2E_1E_2, \tag{66d}$$

where we used the relations in Eqs. (64a)–(64d), together with the identities

$$\sin(\arctan\theta) = \frac{\theta}{\sqrt{1+\theta^2}}, \tag{67a}$$

$$\cos(\arctan\theta) = \frac{1}{\sqrt{1+\theta^2}}. \tag{67b}$$

Notice that the Stokes parameters Eqs. (66a)–(66d) do not depend on the elliptical phase angle δ, but only on the geometric parameters of the ellipse: semimajor axis E_1, seminor axis E_2, and tilt angle $\beta/2$ (Fig. 5), assuming that E_1 is greater than E_2.

The Stokes parameters in the circular basis representation [17,28] can be derived by substituting relations Eqs. (62a), (62b), and (62d) in the Stokes parameters of the rotated ellipse representation in Eqs. (66a)–(66d):

$$S_0 = 2(a_-^2 + a_+^2), \tag{68a}$$

$$S_1 = 4a_+a_-\cos(\delta_+ + \delta_-), \tag{68b}$$

$$S_2 = 4a_+a_-\sin(\delta_+ + \delta_-), \tag{68c}$$

$$S_3 = 2(a_-^2 - a_+^2). \tag{68d}$$

Equations (68a)–(68d) say that the Stokes parameters only depend on the radii a_- and a_+ of the counter-rotating vectors in the circular basis and the sum $\delta_+ + \delta_-$ of the initial phase angles.

B. Field Relation Parameters

To determine the relationship between $\mathbf{E}$ and $\mathbf{H}$ as defined in Eqs. (52a) and (52b), we first take their squares and their products:

$$\mathbf{E}^2 = \zeta^2\mathbf{H}^2 = a_+^2 + a_-^2 + 2a_+a_-\cos(2\phi + \delta_+ + \delta_-), \tag{69a}$$

$$\zeta\mathbf{E}\mathbf{H} = -\zeta\mathbf{H}\mathbf{E} = i\mathbf{e}_3[a_+^2 + a_-^2 + 2a_+a_-\cos(2\phi + \delta_+ + \delta_-)], \tag{69b}$$

where we used the relations (30a) and (30b) and the exponential definition of the cosine in Eq. (20a). Adding and subtracting these squares and products, we get

$$s_0 \equiv \mathbf{E}^2 + \zeta^2\mathbf{H}^2 = 2[a_+^2 + a_-^2 + 2a_+a_-\cos(2\phi + \delta_+ + \delta_-)], \tag{70a}$$

$$s_1 \equiv \mathbf{E}^2 - \zeta^2\mathbf{H}^2 = 0, \tag{70b}$$

$$is_2 \equiv 2i\zeta\mathbf{E}\cdot\mathbf{H} = 0, \tag{70c}$$

$$\mathbf{s}_3 \equiv 2\zeta\mathbf{E}\times\mathbf{H} = 2\mathbf{e}_3[a_+^2 + a_-^2 + 2a_+a_-\cos(2\phi + \delta_+ + \delta_-)], \tag{70d}$$

where s_0, s_1, s_2, and $\mathbf{s}_3$ are what we shall define as the field relation parameters. Equations (70a) and (70d) are new results for elliptically polarized light, but Eqs. (70b) and (70c) are familiar relations for electromagnetic waves. The first relation shows that the energy density $U = (\epsilon_0\mathbf{E}^2 + \mu_0\mathbf{H}^2)/2$ is varying in space and time according to the wave function argument $2\phi = 2(\omega t - kz)$ [8,29]. The second relation shows that the magnitudes of the electric and magnetic fields are proportional, $|\mathbf{E}| = \zeta|\mathbf{H}|$ [34]. The third relation shows that the dot product of the electric and magnetic fields is zero, which means that $\mathbf{E}$ and $\mathbf{H}$ are perpendicular. And the fourth relation shows that the momentum density $\mathbf{S}/c^2 = (\mathbf{E}\times\mathbf{H})/c^2$ is proportional to the energy density $U = |\mathbf{S}|/c$, which corresponds to the well-known relation for light $E = pc$ [35].

Notice that if we set $\phi = 0$, which means $t = 0$ at $z = 0$, then the field relations (70a) and (70d) reduce to

$$s_0|_{\phi=0} = \mathbf{E}^2 + \zeta^2\mathbf{H}^2 = S_0 + S_1, \tag{71a}$$

$$\mathbf{s}_3|_{\phi=0} = 2\zeta\mathbf{E}\times\mathbf{H} = \mathbf{e}_3(S_0 + S_1), \tag{71b}$$

where S_0 and S_1 are the classical Stokes parameters.

Taking the time average over one optical period $\tau = 2\pi/\omega$ of Eqs. (70a)–(70d) as similarly done in Baylis [8], we get

$$\langle s_0\rangle = \langle\mathbf{E}^2 + \zeta^2\mathbf{H}^2\rangle = 2(a_+^2 + a_-^2), \tag{72a}$$

$$\langle s_1\rangle = \langle\mathbf{E}^2 - \zeta^2\mathbf{H}^2\rangle = 0, \tag{72b}$$

$$\langle is_2\rangle = \langle 2i\zeta\mathbf{E}\cdot\mathbf{H}\rangle = 0, \tag{72c}$$

$$\langle\mathbf{s}_3\rangle = \langle 2\zeta\mathbf{E}\times\mathbf{H}\rangle = 2\mathbf{e}_3(a_+^2 + a_-^2). \tag{72d}$$

Notice that the magnitudes of the energy and momentum densities are equal to the first Stokes parameter S_0.

Taking the spatial inverse of the electromagnetic field $\hat{E}$ in Eq. (50), we obtain

$$\hat{E}^\dagger = -\mathbf{E} + i\zeta\mathbf{H}, \tag{73}$$

where $\mathbf{E}$ and $\mathbf{H}$ are defined in Eqs. (52a) and (52b), respectively. It can be shown that the field relation parameters defined in Eqs. (70a)–(70d) are related to the product of the electromagnetic field and its spatial inverse:

$$\hat{E}\hat{E} = s_1 + is_2, \tag{74a}$$

$$\hat{E}^\dagger\hat{E}^\dagger = s_1 - is_2, \tag{74b}$$

$$\hat{E}^\dagger\hat{E} = -s_0 - \mathbf{s}_3, \tag{74c}$$

$$\hat{E}\hat{E}^\dagger = -s_0 - \mathbf{s}_3. \tag{74d}$$

Notice that using the expansions in Eqs. (74a) and (74b), we may combine the relations in Eqs. (72b) and (72c) into $\hat{E}\hat{E} = E^\dagger E^\dagger = 0$.

Now, the product of $\hat{E}^\dagger\hat{E}$ with its spatial inverse can be expressed as

$$(\hat{E}\hat{E}^\dagger)(\hat{E}^\dagger\hat{E}) = (\hat{E}^\dagger\hat{E}^\dagger)(\hat{E}\hat{E}), \tag{75}$$

because the products are associative and $\hat{E}^\dagger\hat{E}^\dagger$ is a scalar by Eq. (74b). Substituting Eqs. (74a)–(74d) into Eq. (75), we get the relation [31]

$$s_0^2 - \mathbf{s}_3^2 = s_1^2 + s_2^2. \tag{76}$$

Equation (76) is similar in form to the Poincaré sphere equation [2,3].

Though the field relation parameters s_0, s_1, is_2, and $\mathbf{s}_3$ in Eqs. (70a)–(70d) satisfy an equation similar to the Poincaré sphere, these parameters are not identical to the Stokes parameters. But even then, the field relation parameters complement the Stokes parameters because the former provides a geometric interpretation for the geometric relations of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$. For monochromatic plane waves that we considered in this paper, the $\mathbf{E}$ and $\mathbf{H}$ are orthogonal in direction and proportional in magnitude to each other so that s_1 and s_2 are zero. But when light is a mixture of several polarization states propagating in different directions, the net electric and magnetic fields of the superposition are not necessarily perpendicular and proportional so that s_1 and s_2 will not be zero. From these two parameters we can determine the relative magnitudes and directions of the time-averaged electric and magnetic fields.

5. CONCLUSION

In this paper, we have presented a geometric algebra formalism for polarized light. We started with Maxwell's equation in free space and proposed an electromagnetic plane wave solution in terms of the product of a circular basis vector with the linear combination of a wave function and its complex conjugate. We transformed the complex-valued wave functions and their coefficients into terms involving the exponentials of an imaginary vector. This transformation allows us to immediately separate the electric and magnetic fields of the electromagnetic wave by taking its vector and imaginary vectors parts.

Because the electric and magnetic fields are similar in form, we only considered the electric field component as done in the literature. This electric field is the sum of two counter-rotating vectors with the exponentials of imaginary vectors acting as vector rotation operators. Using the identities in geometric algebra, we showed that this representation is equivalent to the two other representations: (i) the rotated ellipse representation whose component oscillations are along the semimajor and semiminor axes of the polarization ellipse and (ii) Cartesian basis representations whose component oscillations are along the x and y axes. We computed the relations between the three sets of representation parameters. We also computed the Stokes parameters in the different representations and provided interpretations based on the geometry of the ellipse. We showed that the first Stokes parameter corresponds to the magnitudes of the time-averaged energy and momentum densities. We showed that the electric and magnetic fields are perpendicular in direction and proportional in magnitude. We also defined the field relation parameters in terms of the sum and difference of the squares of the electric and magnetic fields and their dot and cross products. And like the Stokes parameters, they also satisfy an equation similar to the Poincaré sphere equation.

The presentation in this paper of polarized light via geometric algebra has several advantages over the other formalisms. The first advantage is the unifying power of geometric algebra. Geometric algebra unifies complex and vector analyses, the four Maxwell's equations, and the electric and magnetic wave solutions. The second advantage is that we do not disregard imaginary quantities at the end of each computation as is usually done in the literature. Instead, we use imaginary vectors to express rotation operators and to distinguish the magnetic from the electric field. The third advantage is we can use the machinery of geometric algebra to convert

different representations of polarized light. The exponential rotation operators of vectors are used to describe the tilt of the polarization ellipse and to describe its counter-rotating vector components. The fourth advantage is that we define the field relation parameters which describe the geometric relations between the electric and magnetic fields. The vanishing of the second and third field relation parameters means that the electric and magnetic fields are perpendicular to each other and proportional in magnitude. The nonvanishing first and fourth field relation parameters correspond to the electromagnetic energy and momentum densities, whose time averages are proportional to the first Stokes parameter.

In a future work, we shall extend our geometric algebra formalism to optical elements such as rotators, attenuators, phase shifters, polarizers, and birefringent crystals, and discuss the significance of the field relation parameters. We shall also discuss partially polarized light and its relationships to Fourier analysis and Hilbert spaces in geometric algebra.

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Refractive index and energy-momentum of elliptically polarized radio waves in plasma via geometric algebra

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Abstract

In this paper, we revisit the problem of imaginary refractive index in unmagnetized plasma for elliptically polarized light. We verified Budden's statement that the electric and magnetic field of the electromagnetic wave in plasma would be parallel, so that the Poynting vector is zero. But we also showed that the energy-momentum of the wave is not zero: the energy density of the wave is negative and its momentum density is opposite to that if the wave were in a medium with a real refractive index. The formalism that use is the Clifford (geometric) algebra $\mathcal{C}l_{3,0}$, also known as the Pauli algebra.

1 Introduction

It is known that for an isotropic, unmagnetized ionosphere, the square of the refractive index n for an electromagnetic wave with angular frequency ω is given by[1]

$$n^2 = 1 - \frac{Nq^2}{\epsilon m\omega^2}, \tag{1}$$

where N is the ionosphere's electron density, q is the electron's charge, m is the electron's mass, and ϵ_0 is the permittivity of free space.

By the law of Trichotomy, there are only three possibilities: $n^2 > 0$, $n^2 = 0$, and $n^2 < 0$. The interpretation of the first two cases is straightforward, but not the third. Budden analyzed the relations $nE_y = -\mathcal{H}_x$ and $nE_x = \mathcal{H}_y$ and concluded that these hold whether n is real or complex[]Budden 1966 pp 38 41. Furthermore, according to him:

> If n^2 is negative and real, then n is purely imaginary.... This appears to represent a wave travelling with infinite velocity. Every field component varies harmonically in time, but there is no harmonic variation in space. The phase is the same for all values of z. A disturbance of this kind is called an 'evanescent' wave.... In an evanescent wave n is purely imaginary, so that the electric and magnetic fields are in quadrature. This means that the time average value of the Poynting vector $\mathbf{\Pi} = \mathbf{E} \times \mathbf{H}$ is zero. Hence, there is no net energy-flow for such a wave in an isotropic medium. For an evanescent wave in an evanescent medium, there can be some flow of energy perpendicular to the wave-normal.[3]

In this paper, we shall revisit the problem of energy flow for imaginary refractive index. We shall show that even if its the Poynting vector is zero, an elliptically polarized electromagnetic wave would still possess energy and momentum, except that the energy density would be negative and the momentum density would be opposite to that if the refractive index were a real number. We shall also show that for $n = 0$, the energy and momentum densities of the wave would be zero.

We shall divide this paper into six sections. Section 1 is Introduction. In Section 2, we shall discuss axioms and theorems of Clifford (geometric) algebra $\mathcal{C}l_{3,0}$, such as the orthonormality axiom, the Pauli identity, and spatial inversion. We shall also discuss exponential functions and their applications in vector

rotations in the plane. In Section 3, we shall summarize Classical Electrodynamics by discussing the Maxwell's equation, the electromagnetic wave equation, and the electromagnetic energy-momentum . In Section 4, we shall derive the electromagnetic wave equation in plasma by computing the currents due to an elliptically polarized electric field. We then determine the electromagnetic wave equation for plasma and compute the refractive index. We shall find electromagnetic wave solutions to this plasma wave equation and compute their energy-momentum for different cases of refractive indices. Section 5 is Conclusions.

2 Geometric Algebra

2.1 Vectors and Imaginary Numbers

The Clifford (Geometric) algebra $\mathcal{C}l_{3,0}$ is generated by three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ which satisfy the orthonormality axiom[4]

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = \delta_{jk}. \tag{2}$$

In other words,

$$\mathbf{e}_j^2 = 1, \tag{3a}$$

$$\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j, \quad j \neq k. \tag{3b}$$

In other words, the square of a unit vector is unity, while the product of two perpendicular vectors anticommute.

Let us define the unit trivector i:

$$i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. \tag{4}$$

We can show that the trivector i has the following properties:

$$i^2 = -1 \tag{5}$$

and

$$i\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1 i, \tag{6a}$$

$$i\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2 i, \tag{6b}$$

$$i\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3 i, \tag{6c}$$

$$\tag{6d}$$

That is, the trivector i is an imaginary number that commutes with all vectors.

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors spanned by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. We can show that the geometric product of $\mathbf{a}$ and $\mathbf{b}$ is given by the Pauli identity[5]

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b}), \tag{7}$$

where $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are the dot and cross products of the two vectors. From the definition of the geometric product, it is easy to see that

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \tag{8a}$$

$$\mathbf{a}\times\mathbf{b} = \frac{1}{2i}(\mathbf{ab} - \mathbf{ba}). \tag{8b}$$

Thus, the dot product of two vectors is half their anticommutator, while their imaginary cross product is half their commutator.

In general, any object $\hat{A}$ in $\mathcal{C}l_{3,0}$ may be expressed as

$$\hat{A} = A_0 + \mathbf{A}_1 + i\mathbf{A}_2 + iA_3, \tag{9}$$

which we refer to as a cliffor. The spatial inverse $\hat{A}^\dagger$ is obtained by replacing each vector in $\hat{A}^\dagger$ by its negative:

$$\hat{A}^\dagger = A_0 - \mathbf{A}_1 + i\mathbf{A}_2 - iA_3. \tag{10}$$

Notice that under spatial inversion, only the vector and trivector parts change sign; the scalar and bivector parts remain unchanged.

Let $\hat{A}$ and $\hat{B}$ be two cliffors in $\mathcal{C}l_{3,0}$. The spatial inverse of the sum and product of these two cliffors are given by[6]

$$(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger, \tag{11a}$$

$$(\hat{A}\hat{B})^\dagger = \hat{A}^\dagger\hat{B}^\dagger. \tag{11b}$$

In other words, the spatial inversion operator (†) is distributive over addition and multiplication. Despite their similarity in appearance, the spatial inversion operator is not the same as the reversion operator used in the literature, which interchanges the order of vector factors in a cliffor: the reversion operator changes the sign of the bivector and trivector parts, but leaves the scalar and vector parts unchanged.

2.2 Vector Rotations and Exponential Functions

Since $(i\mathbf{e}_3)^2 = -1$, then

$$e^{i\mathbf{e}_3\theta} = \cos\theta + i\mathbf{e}_3 \sin\theta. \tag{12}$$

Left- and right-multiplying Eq. (12) with $\mathbf{e}_1$, $\mathbf{e}_2$, or $\mathbf{e}_3$, we obtain

$$\mathbf{e}_1 e^{i\mathbf{e}_3\theta} = \mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta = e^{-i\mathbf{e}_3\theta}\mathbf{e}_1, \tag{13a}$$
$$\mathbf{e}_2 e^{i\mathbf{e}_3\theta} = -\mathbf{e}_1\sin\theta + \mathbf{e}_2\cos\theta = e^{-i\mathbf{e}_3\theta}\mathbf{e}_2, \tag{13b}$$
$$\mathbf{e}_3 e^{i\mathbf{e}_3\theta} = \mathbf{e}_3\cos\theta + \sin\theta = e^{i\mathbf{e}_3\theta}\mathbf{e}_3. \tag{13c}$$

Geometrically, $\mathbf{e}_1 e^{i\mathbf{e}_3\theta}$ and $\mathbf{e}_2 e^{i\mathbf{e}_3\theta}$ are rotations of $\mathbf{e}_1$ and $\mathbf{e}_2$ about $\mathbf{e}_3$ counterclockwise by an angle θ.

Let us define the vector $\mathbf{s}$ on the $xy-$plane as

$$\mathbf{s} = \mathbf{e}_1 s_1 + \mathbf{e}_2 s_2 = \mathbf{e}_1 s e^{i\mathbf{e}_3\theta}. \tag{14}$$

From this we obtain

$$s = \sqrt{s_1^2 + s_2^2} \tag{15a}$$
$$\theta = \tan^{-1}(s_2/s_1), \tag{15b}$$

and

$$s_1 = s\cos\theta, \tag{16a}$$
$$s_2 = s\sin\theta. \tag{16b}$$

Equations (15a) to (16b) are the familiar transformation relations for the polar and rectangular coordinates.

The vector $\mathbf{s}$ in Eq. (14) may also be written as[7]

$$\mathbf{s} = \mathbf{e}_1\hat{s} = \hat{s}^*\mathbf{e}_1, \tag{17}$$

where

$$\hat{s} = s_1 + i\mathbf{e}_3 s_2 = s e^{i\mathbf{e}_3\theta}, \tag{18a}$$
$$\hat{s}^* = s_1 - i\mathbf{e}_3 s_2 = s e^{-i\mathbf{e}_3\theta}. \tag{18b}$$

We say that $\hat{s}$ is the complex representation of the vector $\mathbf{s}$ and $\hat{s}^*$ is the complex conjugate of $\hat{s}$.

Let $\mathbf{s}'$ be another vector in the $xy-$plane:

$$\mathbf{s} = \mathbf{e}_1 s_1 + \mathbf{e}_2 s_2 = \mathbf{e}_1 s e^{i\mathbf{e}_3\theta}. \tag{19}$$

The product of $\mathbf{s}$ and $\mathbf{s}'$ is

$$\mathbf{s}\mathbf{s}' = \mathbf{e}_1\hat{s}\mathbf{e}_1\hat{s}' = \hat{s}^*\hat{s}' = ss'e^{i\mathbf{e}_3(-\theta+\theta')}. \tag{20}$$

Using the Pauli identity in Eq. (7) together with the Euler's formula in Eq. (12), and separating the scalar and bivector parts of Eq. (20), we arrive at

$$\mathbf{s}\cdot\mathbf{s}' = s_1 s_1' + s_2 s_2' = ss'\cos(-\theta+\theta'), \tag{21a}$$
$$\mathbf{s}\times\mathbf{s}' = i\mathbf{e}_3(s_1 s_2' - s_1' s_2) = i\mathbf{e}_3 ss'\sin(-\theta+\theta'), \tag{21b}$$

which are the familiar relations for the dot and cross products of two dimensional vectors in rectangular and polar forms.

3 Classical Electrodynamics

3.1 Maxwell's Equation

In a medium characterized by dielectric permittivity ϵ_0 and magnetic permeability μ_0, the Maxwell's equation in electrodynamics is given by

$$\frac{\partial\hat{E}}{\partial\hat{r}} = \zeta_0\hat{j}^\dagger, \tag{22}$$

where

$$\frac{\partial}{\partial\hat{r}} = \frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\nabla}, \tag{23a}$$
$$\hat{E} = \mathbf{E} + i\zeta_0\mathbf{H}, \tag{23b}$$
$$\hat{j} = \rho c + \mathbf{j}. \tag{23c}$$

Note that the speed of light $c = 1/\sqrt{\mu_0\epsilon_0}$ and the intrinsic impedance $\zeta_0 = \sqrt{\mu_0/\epsilon_0}$. In words, the Maxwell's equation states that the spacetime derivative $\partial/\partial\hat{r}$ of the electromagnetic field $\hat{E}$ is proportional to the spatial inverse of the charge-current density $\hat{j}$.

Let us expand the Maxwell's Equation in Eq. (22):[8, 9]

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\nabla}\right)(\mathbf{E} + i\zeta_0\mathbf{H}) = \zeta_0(\rho c - \mathbf{j}). \tag{24}$$

Using the Pauli identity in Eq. (7) and separating the scalar, vector, bivector, and trivector parts, we arrive at

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{25a}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \zeta_0 \nabla \times \mathbf{H} = -\zeta_0 \mathbf{j}, \tag{25b}$$

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{25c}$$

$$\zeta_0 \nabla \cdot \mathbf{H} = 0, \tag{25d}$$

which are the Gauss's law, Ampere's law, Faraday's law, and magnetic flux continuity law, respectively.

3.2 Wave Equation

The spatial inverse of the spacetime derivative operator is

$$\frac{\partial}{\partial \hat{r}^\dagger} = \frac{1}{c}\frac{\partial}{\partial t} - \nabla. \tag{26}$$

Applying this operator to the Maxwell's equation in Eq. (22), we obtain

$$\Box \hat{E} = \zeta_0 \frac{\partial \hat{j}^\dagger}{\partial \hat{r}^\dagger}. \tag{27}$$

where

$$\Box = \frac{\partial^2}{\partial \hat{r}^\dagger \partial \hat{r}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{28}$$

is the d'Alembertian operator.

Expanding Eq. (27),

$$\Box(\mathbf{E} + i\zeta_0 \mathbf{H}) = \zeta_0 \left(\frac{1}{c}\frac{\partial}{\partial t} - \nabla\right)(\rho c - \mathbf{j}), \tag{29}$$

and separating the scalar, vector, and bivector parts, we get

$$0 = \frac{\partial \rho}{\partial t} - \nabla \cdot \mathbf{j}, \tag{30a}$$

$$\Box \mathbf{E} = -\frac{1}{\epsilon_0}\nabla \rho - \mu_0 \frac{\partial \mathbf{j}}{\partial t}, \tag{30b}$$

$$\Box \mathbf{H} = \nabla \times \mathbf{j}. \tag{30c}$$

The first equation is the continuity condition[10], while the second and third are the wave equations for the electric and magnetic field intensities[11], respectively.

If the charge distribution ρ is nearly constant, then

$$\nabla \rho = 0, \tag{31}$$

so that the wave equation in Eq. (30b) simplifies to

$$\Box \mathbf{E} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t}. \tag{32}$$

Later we shall express the current density $\mathbf{j}$ in terms of the electric field intensity $\mathbf{E}$, in order to determine the refractive index and energy-momentum of the electromagnetic wave in plasma.

3.3 Energy-Momentum

The product of the electromagnetic field $\hat{E} = \mathbf{E} + i\zeta_0 \mathbf{H}$ and its spatial inverse $\hat{E}^\dagger = -\mathbf{E} + i\zeta \mathbf{H}$ is

$$\hat{E}\hat{E}^\dagger = (\mathbf{E} + i\zeta_0 \mathbf{H})(-\mathbf{E} + i\zeta_0 \mathbf{H}), \tag{33}$$

which may be expanded as[5, 6]

$$\hat{E}\hat{E}^\dagger = -|\mathbf{E}|^2 - \zeta_0^2 |\mathbf{H}|^2 - 2\zeta_0 \mathbf{E} \times \mathbf{H}. \tag{34}$$

This expansion lets us to define the energy-momentum cliffor $\hat{S}/c$:

$$\frac{\hat{S}}{c} = -\frac{1}{2}\epsilon_0 \hat{E}\hat{E}^\dagger = U + \frac{\mathbf{S}}{c}, \tag{35}$$

where

$$U = \frac{1}{2}(\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2), \tag{36a}$$

$$\frac{S}{c^2} = \frac{\mathbf{E} \times \mathbf{H}}{c^2} \tag{36b}$$

are the energy and momentum densities, respectively. The vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector.

4 Wave Equation in Plasma

4.1 Electron Orbits

If we assume that the only force acting on the charges is significantly due to the electric field of the electromagnetic waves, then we may write Newton's laws of motion as [12]

$$m\frac{\partial \mathbf{r}}{\partial t^2} = q\mathbf{E}. \tag{37}$$

Note that we assume that the acceleration of the electron is mainly due to the electric field $\mathbf{E}$ of the electromagnetic wave and contribution of the wave's magnetic field to the electron's motion is negligible.

In general, the electromagnetic wave is ellipticaly polarized, so that its electric field may be expressed as:[13]

$$\mathbf{E} = \mathbf{e}_1(\hat{a}_-\hat{\psi} + \hat{a}_+\hat{\psi}) \tag{38}$$

where

$$\hat{a}_\pm = \hat{a}_\pm e^{i\mathbf{e}_3\alpha_\pm}, \tag{39a}$$

$$\hat{\psi}^{\pm 1} = e^{\pm i\mathbf{e}_3(\omega t - kz)} \tag{39b}$$

are the complex amplitudes and the wave functions. Using the exponential relation in Eq. (13a), Eq. (38) may be expanded as

$$\begin{aligned}\mathbf{E} = \mathbf{e}_1(&a_-\cos(\omega t - kz - \alpha_-)+\\ &a_+\cos(\omega t - kz + \alpha_+))+\\ \mathbf{e}_2(&-a_-\sin(\omega t - kz - \alpha_-)+\\ &a_+\sin(\omega t - kz + \alpha_+)).\end{aligned} \tag{40}$$

Though Eq. (40) is more explicit than Eq. (38), the expression involving the wave functions is easier to interpret geometrically: the electric field $\mathbf{E}$ is a sum of two counter-rotating circular motions. The circles have magnitudes a_- and a_+, and phases α_- and α_+. For fixed position z, the rotations are counterclockwise for angular frequency $\omega > 0$ and clockwise for angular frequency $\omega < 0$. For fixed time t, the rotations are clockwise for wave number $k > 0$ and clockwise for wave number $k < 0$.

To solve equation of motion in Eq. (37), we assume that the total solution $\mathbf{r}$ may be written as solutions:

$$\mathbf{r} = \mathbf{r}_h + \mathbf{r}_p, \tag{41}$$

where $\mathbf{r}_h$ is the homogeneous solution and $\mathbf{r}_p$ is the particular solution.

The homogeneous solution $\mathbf{r}_h$ satisfies the homogeneous equation

$$m\frac{\partial^2 \mathbf{r}_h}{\partial t^2} = 0. \tag{42}$$

From this we see that the solution is a linear motion:

$$\mathbf{r}_h = \mathbf{r}'_0 + \mathbf{v}'_0 t, \tag{43}$$

where $\mathbf{r}'_0$ and $\mathbf{v}'_0$ are unknown vector parameters that still have to be determined from the boundary conditions.

On the other hand, for the equation for the particular solution is

$$\frac{\partial^2 \mathbf{r}_p}{\partial t^2} = q\mathbf{E}. \tag{44}$$

If we assume that the position $\mathbf{r}$ of the electron is proportional to the forcing electric field $\mathbf{E}$ in Eq. (38), then

$$\mathbf{r}_p = \gamma\mathbf{E}, \tag{45}$$

where γ is a constant. Since the time derivatives of the electric field $\mathbf{E}$ in Eq. (38) are

$$\frac{\partial \mathbf{E}}{\partial t} = \mathbf{e}_1 i\mathbf{e}_3\omega(-\hat{a}_-\hat{\psi}^{-1} + \hat{a}_+\hat{\psi}), \tag{46a}$$

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{e}_1(i\mathbf{e}_3\omega)^2(\hat{a}_-\hat{\psi}^{-1} + \hat{a}_+\hat{\psi}) = -\omega^2\mathbf{E}. \tag{46b}$$

So substituting the assumed solution in Eq. (45) and using the expression for the second time derivative of the electric field $\mathbf{E}$ in Eq. (46b), we arrive at

$$\gamma = -\frac{q}{m\omega^2}. \tag{47}$$

Since $\gamma < 0$, then the particular solution $\mathbf{r}_p$ is opposite in direction to the forcing electric field $\mathbf{E}$.

Substituting the results for $\mathbf{r}_h$ and $\mathbf{r}_p$ in Eqs. (43) and (45) in Eq. (41), we obtain

$$\mathbf{r} = \mathbf{r}'_0 + \mathbf{v}'_0 t - \frac{q}{m\omega^2}\mathbf{E}, \tag{48}$$

Equation (48) is the total solution for the electron's orbit under the electric field $\mathbf{E}$. Notice that the motion of the electron is a sum of a linear motion $\mathbf{r}'_0 + \mathbf{v}'_0 t$ and an elliptical motion opposite to the electric field $\mathbf{E}$.

Let us assume that at $t = 0$, the electron is stationary, so that

$$\mathbf{r} = \mathbf{r}'_0, \tag{49a}$$

$$\mathbf{v} = 0. \tag{49b}$$

With these boundary conditions, Eq. (48) leads to two simultaneous equations:

$$\mathbf{r}_0 = \mathbf{r}'_0 - \frac{q}{m\omega^2}\mathbf{e}_1(\hat{a}_- + \hat{a}_+), \tag{50a}$$

$$0 = \mathbf{v}'_0 - \frac{q}{m\omega}\mathbf{e}_1 i\mathbf{e}_3(-\hat{a}_- + \hat{a}_+). \tag{50b}$$

Solving for $\mathbf{r}_0'$ and $\mathbf{v}_0'$, we get

$$\mathbf{r}_0' = \mathbf{r}_0 + \frac{q}{m\omega^2}\mathbf{e}_1(\hat{a}_- + \hat{a}_+), \tag{51a}$$

$$\mathbf{v}_0' = \frac{q}{m\omega}\mathbf{e}_2(-\hat{a}_- + \hat{a}_+). \tag{51b}$$

Equations (51a) and (51b) are the expressions for the unknown vector parameters of the charge's position in time in terms of the charge's initial position $\mathbf{r}_0$ and initial velocity $\mathbf{v}_0$.

Substituting Eqs. (51a) and (51b) back to the total solution in Eq. (48), we obtain

$$\mathbf{r} = \mathbf{r}_0 + \frac{q}{m\omega^2}\mathbf{e}_1(\hat{a}_- + \hat{a}_+) + \frac{q}{m\omega}\mathbf{e}_2(-\hat{a}_- + \hat{a}_+)t - \frac{q}{m\omega^2}\mathbf{e}_1(\hat{a}_-\hat{\psi}^{-1} + \hat{a}_+\hat{\psi}). \tag{52}$$

Using the exponential relation in Eq. (13a), Eq. (52) expands to

$$\begin{aligned}\mathbf{r} = \mathbf{e}_1(&x_0 + a_- \cos\alpha_- + a_+ \cos\alpha_+ + \\ &\frac{q}{m\omega^2}(((a_- \sin\alpha_- - a_+ \sin\alpha_+)t \\ &\quad - a_- \cos(\omega t - kz - \alpha_-) \\ &\quad - a_+ \cos(\omega t - kz + \alpha_+))) + \\ \mathbf{e}_2(&y_0 + a_- \sin\alpha_- + a_+ \sin\alpha_+ + \\ &\frac{q}{m\omega^2}((-a_- \sin\alpha_- + a_+ \sin\alpha_+)t \\ &\quad + a_- \sin(\omega t - kz - \alpha_-) \\ &\quad - a_+ \sin(\omega t - kz + \alpha_+))).\end{aligned} \tag{53}$$

Equation (53) gives the orbit of the electron which is initially motionless prior to its interaction with an elliptically polarized electromagnetic wave of frequency ω.

4.2 Wave Equation

4.2.1 Electric Field

To obtain the wave equation for the electric field $\mathbf{E}$, we assume that the ionosphere is made up of electrons of charge q. If N is the number of charges per unit volume and these charges are moving with velocity $\mathbf{v}$, then the current density $\mathbf{j}$ is given by

$$\mathbf{j} = Nq\frac{\partial \mathbf{r}}{\partial t}. \tag{54}$$

Using the expression for the total solution $\mathbf{r}$ in Eq. (52), together with the time derivative of the electric field $\mathbf{E}$ in Eq. (46a), we arrive at

$$\mathbf{j} = Nq\left(\mathbf{v}_0' - \frac{q}{m\omega^2}\frac{\partial \mathbf{E}}{\partial t}\right) \tag{55}$$

Notice that the current density is a sum of an constant current density and the time derivative of the electric field of the electromagnetic wave.

Substituting the expression for the current density $\mathbf{j}$ in Eq. (55) to the wave equation in Eq. (32), we get

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{E} = \mu_0\frac{Nq^2}{m\omega^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{56}$$

Using the expression $c = 1/\sqrt{\mu_0\epsilon_0}$, we may rewrite Eq. (56) as

$$\frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{E} = 0, \tag{57}$$

where

$$n = \sqrt{1 - \frac{Nq^2}{\epsilon_0 m\omega^2}} \tag{58}$$

is the plasma refractive index[1].

Equation (58) suggests the following definition for plasma frequency:

$$\omega_0 = \frac{Nq^2}{\epsilon_0 m}. \tag{59}$$

In this way, the refractive index n in Eq. (58) simplifies to

$$n = \sqrt{1 - \frac{\omega_0^2}{\omega^2}}. \tag{60}$$

Notice that at high electromagnetic wave frequencies, $\omega >> \omega_0$, the ionosphere's refractive index approaches unity. When the electromagnetic frequencies approaches the plasma frequency, $\omega \to \omega_0$, the refractive index becomes zero. And below the plasma frequency, $\omega < \omega_0$, the refractive index is imaginary.

4.2.2 Magnetic Field

On the other hand, to obtain the wave equation for the magnetic field $\mathbf{H}$, we take the spatial derivative $\boldsymbol{\nabla}$ of the current density $\mathbf{j}$ in Eq. (55) to obtain

$$\boldsymbol{\nabla}\mathbf{j} = -\frac{Nq^2}{m\omega^2}\frac{\partial}{\partial t}\boldsymbol{\nabla}\mathbf{E}. \tag{61}$$

Separating the scalar and bivector parts, we get

$$\nabla \cdot \mathbf{j} = -\frac{Nq^2}{m\omega^2}\frac{\partial}{\partial t}\nabla \cdot \mathbf{E}, \tag{62a}$$

$$\nabla \times \mathbf{j} = -\frac{Nq^2}{m\omega^2}\frac{\partial}{\partial t}\nabla \times \mathbf{E}, \tag{62b}$$

after factoring out i in the second equation.

Using Faraday's law in Eq. (25c), Eq. (62b) becomes

$$\nabla \times \mathbf{j} = \frac{Nq^2\mu_0}{m\omega^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{63}$$

Substituting this to the wave equation for the magnetic field in Eq. (30c) and rearranging the terms, we get

$$\frac{n^2}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} - \nabla^2 \mathbf{H} = 0, \tag{64}$$

where we used the definition of the refractive index n in Eq. (58). Equation (64) is the magnetic field analog of the plasma wave equation for the electric field $\mathbf{E}$ in Eq. (57).

4.2.3 Electromagnetic Field

In our unperturbed field approximation, we used the initial electric field $\mathbf{E}$ to compute the current density $\mathbf{j}$ to obtain the current density of the medium as a function of the original electric field. Clearly, the original electric and magnetic fields do not satisfy the new wave equations in Eqs. (57) and (64). To reflect this change, let us use primed variables for the new electric and magnetic fields in plasma, so that the new wave equations become:

$$\Box' \mathbf{E}' = 0, \tag{65a}$$

$$\Box' \mathbf{H}' = 0, \tag{65b}$$

where

$$\Box' = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{66}$$

is the d'Alembertian operator in a medium with refractive index n.

To combine the electric and magnetic wave equations into one, we need to compute first the plasma impedance ζ in terms of the plasma electric permittivity ϵ and plasma magnetic permeability μ. For the magnetic permeability μ, let us assume that it is equal to vacuum permeability μ_0, since the magnetic field $\mathbf{H}$ of the electromagnetic wave has negligible effect on the motion of the charges compared to that of the electric field:

$$\mu = \mu_0. \tag{67}$$

For the permittivity ϵ, we know that the wave speed c/n is related to the permittivity and permeability by

$$\frac{c}{n} = \frac{1}{\sqrt{\mu\epsilon}}. \tag{68}$$

Solving for the permittivity ϵ and using the result in Eq. (67), we get

$$\epsilon = n^2\epsilon_0, \tag{69}$$

where we used the relation $c = 1/\sqrt{\mu_0\epsilon_0}$. Thus, the ratio of the permittivity of the plasma and vacuum is the square of the refractive index n.

Since we already know the plasma permeability μ in Eq. (67) and the plasma permittivity ϵ in Eq. (69), we can now compute the plasma impedance ζ:

$$\zeta = \sqrt{\frac{\mu}{\epsilon}} = \frac{1}{n}\zeta_0. \tag{70}$$

Thus, the ratio of the impedances of the plasma and vacuum is equal to the reciprocal of the refractive index n.

We may now construct our electromagnetic field $\hat{E}$ as

$$\hat{E}' = \mathbf{E}' + i\zeta\mathbf{H}', \tag{71}$$

so that the two wave equations in Eq. (65a) and (65b) combines to

$$\Box'(\mathbf{E}' + i\frac{\zeta_0}{n}\mathbf{H}') = 0. \tag{72}$$

Equation (72) is now a homogeneous electromagnetic wave equation in a plasma with refractive index n.

4.3 EM Wave

But there are difficulties. If we use the expression for the current density $\mathbf{j}$ in Eq. (55) in the Ampere's law in Eq. (25b) , we get

$$\frac{1}{c}\frac{\partial \mathbf{E}'}{\partial t} - \zeta_0 \nabla \times \mathbf{H}' = -\zeta_0 Nq\left(\mathbf{v}'_0 - \frac{q}{m\omega^2}\frac{\partial \mathbf{E}'}{\partial t}\right), \tag{73}$$

where we already replaced the fields by their primed counterparts. Rearranging the terms, we have

$$\frac{n^2}{c}\frac{\partial \mathbf{E}'}{\partial t} - \zeta_0 \boldsymbol{\nabla} \times \mathbf{H}' = -\zeta_0 N q \mathbf{v}_0'. \qquad (74)$$

It is possible that this extra current density on the right-hand side is an artifact of our not considering the magnetic field of the electromagnetic wave in the Lorentz force when we computed the orbit of the charges Eq. (37) from which we obtained the current density. The magnetic field would provide the necessary reaction term that would prevent the charges from drifting away. In this paper, we shall simply drop the $\mathbf{v}_0'$ term in the Ampere's law:

$$\frac{n^2}{c}\frac{\partial \mathbf{E}'}{\partial t} - \zeta_0 \boldsymbol{\nabla} \times \mathbf{H}' = 0. \qquad (75)$$

We shall also assume that the net charge density is low. That is,

$$\rho \approx 0. \qquad (76)$$

This may be a good approximation in the ionosphere, since the electrons are knocked out from the atoms, so that the effective charge density is zero. The positive ions would be more massive than the electrons, so that the positive ions would move much more slowly than the electrons. Hence, the polarizability of the ionosphere would be mainly due to the electrons.

Equations (74) and (76) justify the following approximation:

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \boldsymbol{\nabla}\right)(\mathbf{E}' + i\frac{\zeta_0}{n}\mathbf{H}') = 0, \qquad (77)$$

which is the homogeneous Maxwell's equation in plasma. This means that we may use the form of the electromagnetic wave in vacuum as our trial solution[13]:

$$\hat{E}' = \mathbf{E}' + i\frac{1}{n}\zeta_0 \mathbf{H}' = \hat{\mathbf{e}}_+(\hat{a}_-'\hat{\psi}'^{-1} + \hat{a}_+'\hat{\psi}'), \qquad (78)$$

where

$$\hat{\mathbf{e}}_+ = \mathbf{e}_1 + i\mathbf{e}_2, \qquad (79a)$$

$$\hat{\psi}' = e^{i\mathbf{e}_3(\omega t - k'z)} \qquad (79b)$$

is complex basis vector and the plasma wave function.

To determine the wave number k' in Eq. (79b), we substitute Eq. (78) back to the wave equation in Eq. (72), we obtain

$$\left(\frac{n^2\omega^2}{c^2} - k'^2\right)\hat{E} = 0. \qquad (80)$$

For this equation to be satisfied, the first factor must be zero, so that

$$k' = nk, \qquad (81)$$

where $k = \omega/c$. Thus, the ratio of the wave number k' in plasma and that of k in vacuum is the refractive index n.

Let us consider three cases: $n > 0$, $n = 0$, and n is imaginary.

4.3.1 Case 1: $n^2 > 0$

If $n^2 > 0$, then the electromagnetic wave $\hat{E}'$ in Eq. (78) is simply an elliptically polarized wave which oscillates in both time and space. Separating the vector and bivector parts of Eq. (78), we get

$$\mathbf{E} = \mathbf{e}_1(\hat{a}_-'\hat{\psi}'^{-1} + \hat{a}_+\hat{\psi}'), \qquad (82a)$$

$$\mathbf{H} = \mathbf{e}_2 n(\hat{a}_-'\hat{\psi}'^{-1} + \hat{a}_+\hat{\psi}'). \qquad (82b)$$

These may be expanded as

$$\begin{aligned}\mathbf{E}' = \mathbf{e}_1(&a_-'\cos(\omega t - nkz - \alpha_-') \\ &+ a_+'\cos(\omega t - nkz + \alpha_+'))+ \\ \mathbf{e}_2(&-a_-'\sin(\omega t - nkz - \alpha_-') \\ &+ a_+'\sin(\omega t - nkz + \alpha_+')), \qquad (83a)\\ \mathbf{H}' = \mathbf{e}_2 n(&a_-'\cos(\omega t - nkz - \alpha_-') \\ &+ a_+'\cos(\omega t - nkz + \alpha_+'))+ \\ \mathbf{e}_1 n(&a_-'\sin(\omega t - nkz - \alpha_-') \\ &- a_+'\sin(\omega t - nkz + \alpha_+')). \qquad (83b)\end{aligned}$$

Notice that the electric and magnetic fields are both elliptically polarized since they are both a sum of the same counter-rotating. The only difference is that the two fields are perpendicular to each other.

4.3.2 Case 2: $n^2 < 0$

And if $n^2 < 0$, then we have to decide on which unit imaginary number should we use, because there are at least four distinct objects which square to -1: i, $i\mathbf{e}_1$, $i\mathbf{e}_2$, and $i\mathbf{e}_3$. From the form of the wave function $\hat{\psi}^+-'$ in Eq. (79b), we see that the proper unit imaginary number should be $i\mathbf{e}_3$, so that

$$n = \pm i\mathbf{e}_3|n|, \tag{84}$$

where

$$|n| = \sqrt{\left|1 - \frac{Nq^2}{\epsilon_0 m\omega^2}\right|}. \tag{85}$$

is the magnitude of the refractive index.

Using the expressions in Eqs. (84) and (81), the electromagnetic wave in Eq. (78) becomes

$$\mathbf{E}'_\pm \pm \frac{1}{|n|}\zeta_0\mathbf{e}_3\mathbf{H}'_\pm = \hat{\mathbf{e}}_+(\hat{a}'_- e^{-i\mathbf{e}_3\omega t}e^{\mp|n|kz} + \hat{a}'_+ e^{i\mathbf{e}_3\omega t}e^{\pm|n|kz}), \tag{86}$$

where we used the identities

$$(i\mathbf{e}_3)^{-1} = \frac{1}{i\mathbf{e}_3} = -i\mathbf{e}_3, \tag{87a}$$

$$(\mathbf{e}_3)^2 = 1. \tag{87b}$$

If we assume that $\mathbf{H}'_\pm$ is in the $xy-$plane, then $\mathbf{e}_3 \cdot \mathbf{H}_\pm = 0$, so that

$$\mathbf{e}_3\mathbf{H}_\pm = i(\mathbf{e}_3 \times \mathbf{H}_\pm) = -\mathbf{H}_\pm\mathbf{e}_3. \tag{88}$$

is a bivector. This result lets us separate the vector and bivector parts of Eq. (86) into

$$\mathbf{E}'_\pm = \mathbf{e}_1(\hat{a}'_- e^{-i\mathbf{e}_3\omega t}e^{\mp|n|kz} + \hat{a}'_+ e^{i\mathbf{e}_3\omega t}e^{\pm|n|kz}), \tag{89a}$$

$$\zeta_0\mathbf{H}'_\pm = \pm\mathbf{e}_1|n|(\hat{a}'_- e^{-i\mathbf{e}_3\omega t}e^{\mp|n|kz} + \hat{a}'_+ e^{i\mathbf{e}_3\omega t}e^{\pm|n|kz}), \tag{89b}$$

where we multiplied the bivector part by $\pm\mathbf{e}_3$ and used the identity

$$\mathbf{e}_3 i\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_1. \tag{90}$$

Expanding the exponentials, Eqs. (89a) and (89b) become

$$\begin{aligned}\mathbf{E}'_\pm = \mathbf{e}_1(&a'_- e^{\mp|n|kz}\cos(\omega t - \alpha'_-) + \\ &a'_+ e^{\pm|n|kz}\cos(\omega t + \alpha'_-)) + \\ &\mathbf{e}_2(-a'_- e^{\mp|n|kz}\sin(\omega t - \alpha'_-) + \\ &a'_+ e^{\pm|n|kz}\sin(\omega t + \alpha'_-)),\end{aligned} \tag{91a}$$

$$\begin{aligned}\zeta_0\mathbf{H}'_\pm = \pm\mathbf{e}_1|n|(&a'_- e^{\mp|n|kz}\cos(\omega t - \alpha'_-) + \\ &a'_+ e^{\pm|n|kz}\cos(\omega t + \alpha'_-)) + \\ &\pm\mathbf{e}_2|n|(-a'_- e^{\mp|n|kz}\sin(\omega t - \alpha'_-) \\ &a'_+ e^{\pm|n|kz}\sin(\omega t + \alpha'_-)).\end{aligned} \tag{91b}$$

Notice that $\mathbf{E}_\pm \parallel \mathbf{H}_\pm$, which agrees with Budden.

In the limit as $z >> 1/|n|k$, the terms involving $e^{-|n|kz} \to 0$, so that Eqs. (89a) and (89b) reduces to

$$\mathbf{E}'_\pm = \mathbf{e}_1\hat{a}'_\pm e^{|n|kz}e^{\pm i\mathbf{e}_3\omega t}, \tag{92a}$$

$$\zeta_0\mathbf{H}'_\pm = \pm\mathbf{e}_1|n|\hat{a}'_\pm e^{|n|kz}e^{\pm i\mathbf{e}_3\omega t}. \tag{92b}$$

Expanding the exponentials, we get

$$\mathbf{E}'_\pm = a_\pm e^{|n|kz}(\mathbf{e}_1\cos(\omega t \pm \alpha'_\pm) \pm \mathbf{e}_2\sin(\omega t \pm \alpha'_\pm)), \tag{93a}$$

$$\zeta_0\mathbf{H}'_\pm = \pm a_\pm e^{|n|kz}(\mathbf{e}_1\cos(\omega t \pm \alpha'_\pm) \pm \mathbf{e}_2\sin(\omega t \pm \alpha'_\pm)). \tag{93b}$$

In other words, as the propagation distance z becomes very large, the electric and magnetic fields become circularly polarized and their amplitudes increase exponentially with distance.

4.3.3 Case 3: $n^2 = 0$

If $n^2 \to 0^+$, then Eq. (78) reduces to

$$\hat{E}' = \mathbf{E}' + i\frac{1}{n}\zeta_0\mathbf{H}' = (\mathbf{e}_1 + i\mathbf{e}_2)(\hat{a}'_- e^{-i\mathbf{e}_3\omega t} + \hat{a}'_+ e^{i\mathbf{e}_3\omega t}). \tag{94}$$

Separating the scalar and bivector parts,

$$\mathbf{E}' = \mathbf{e}_1(\hat{a}'_- e^{-i\mathbf{e}_3\omega t} + \hat{a}'_+ e^{i\mathbf{e}_3\omega t}), \tag{95a}$$

$$\zeta_0\mathbf{H}' = 0, \tag{95b}$$

so that only the electric field remains. Expanding the expression for the electric field, we arrive at

$$\mathbf{E}' = \mathbf{e}_1(a_- \cos(\omega t - \alpha'_-) + a_+ \cos(\omega t + \phi'_+)) + \\ \mathbf{e}_2(-a_- \sin(\omega t - \alpha_-) + a_+ \sin(\omega t + \phi_+)). \tag{96}$$

Geometrically, as $n^2 \to 0^+$, the electric field $\mathbf{E}'$ becomes a sum of two counter-rotating circular motions with frequency ω. The circles have radii a_- and a_+, and phase angles α'_- and α'_+. The rotations does not change phase as the position z of the wave varies.

On the other hand, if $n^2 \to 0^-$, then Eq. (89a) and (89b) reduces to

$$\mathbf{E}'_\pm = \mathbf{e}_1(\hat{a}'_- e^{-i\mathbf{e}_3\omega t} + \hat{a}'_+ e^{i\mathbf{e}_3\omega t}), \tag{97a}$$

$$\zeta_0 \mathbf{H}'_\pm = 0, \tag{97b}$$

which are the same equations given in Eqs. (95a) and (95b).

4.4 Wave Energy-Momentum

The energy-momentum of an electromagnetic field in plasma is is analogous to that in vacuum as given in Eq. (35):

$$-\frac{1}{2}\epsilon \hat{E}' \hat{E}'^\dagger = U' + \sqrt{\mu\epsilon}\,\mathbf{S}, \tag{98}$$

where

$$U' = \frac{1}{2}(\epsilon \mathbf{E}'^2 + \mu \mathbf{H}'^2), \tag{99a}$$

$$\mu\epsilon \mathbf{S}' = \mu\epsilon \mathbf{E}' \times \mathbf{H}', \tag{99b}$$

In terms of the refractive index n and the speed of light in vacuum c, Eq. (98) may be written as

$$-\frac{1}{2}\epsilon \hat{E}' \hat{E}'^\dagger = U' + \frac{n}{c}\mathbf{S}', \tag{100}$$

where

$$U' = \frac{1}{2}(\epsilon_0 n^2 \mathbf{E}'^2 + \mu_0 \mathbf{H}'^2), \tag{101a}$$

$$\frac{n^2}{c^2}\mathbf{S}' = \frac{n^2}{c^2}\mathbf{E}' \times \mathbf{H}', \tag{101b}$$

where we used the relations $\mu = \mu_0$ and $\epsilon = n^2\epsilon_0$ in Eqs. (67) and (69), respectively. Eqs. (101a) and (101b) are the energy and momentum densities of the electromagnetic field in plasma.

Let us compute the energy-momentum of the electromagnetic wave $\hat{E}'$ in Eq. (78), by considering three cases: $n^2 > 0$, $n^2 = 0$, and $n^2 < 0$.

4.4.1 Case 1: $n^2 > 0$

For $n^2 > 0$, the energy momentum of the electromagnetic wave $\hat{E}'$ in Eq. (78) is

$$U' + \frac{n}{c}\mathbf{S}' = -\frac{1}{2}\epsilon_0 n^2 \hat{\mathbf{e}}_+ (\hat{a}'_- \hat{\psi}'^{-1} + \hat{a}'_+ \hat{\psi}') \\ \hat{\mathbf{e}}_+^\dagger (\hat{a}'_- \hat{\psi}'^{-1} + \hat{a}'_+ \hat{\psi}'), \tag{102}$$

since

$$\hat{a}'^\dagger_\pm = \hat{a}'_\pm, \tag{103a}$$

$$(\hat{\psi}'^{\pm 1})^\dagger = \hat{\psi}'^{\pm 1}, \tag{103b}$$

because scalar-bivector cliffors are invariant under spatial inversion.

From the relations in Eqs. (13a) and (13b), we have

$$\hat{\mathbf{e}}_+ \psi^{\pm 1} = \hat{\psi}^{\mp 1} \hat{\mathbf{e}}_+, \tag{104a}$$

$$\hat{\mathbf{e}}_+ \hat{a}_\pm = \hat{a}^*_\pm \hat{\mathbf{e}}_+. \tag{104b}$$

Hence, Eq. (102) becomes

$$U' + \frac{n}{c}\mathbf{S}' = -\frac{1}{2}\epsilon_0 n^2 \hat{\mathbf{e}}_+ \hat{\mathbf{e}}_+^\dagger (\hat{a}'^*_- \hat{\psi}' + \hat{a}'^*_+ \hat{\psi}'^{-1}) \\ (\hat{a}'_- \hat{\psi}'^{-1} + \hat{a}'_+ \hat{\psi}'), \tag{105}$$

Distributing the terms in Eq. (105) and using the identity

$$\hat{\mathbf{e}}_+ \hat{\mathbf{e}}_+^\dagger = -2(1 + \mathbf{e}_3) \tag{106}$$

we obtain

$$U' + \frac{n}{c}\mathbf{S}' = \frac{1}{2}\epsilon_0 n^2 (1 + \mathbf{e}_3)(a_-^2 + a_+^2 \\ + \hat{a}'^\dagger_- \hat{a}'_+ \hat{\psi}'^2 + \hat{a}'^\dagger_+ \hat{a}'_- \hat{\psi}'^{-1}). \tag{107}$$

Using the definitions of Eq. (39a) and (79b), Eq. (107) becomes

$$U' + \frac{n}{c}\mathbf{S}' = (1 + \mathbf{e}_3)\epsilon_0 n^2 (a'^2_- + a'^2_+ + 2a'_- a'_+ \times \\ \cos(2(\omega t - nkz) + \alpha'_+ - \alpha'_-)). \tag{108}$$

Separating the scalar and vector parts of Eq. (108), we get

$$U' = \epsilon_0 n^2 (a_-'^2 + a_+'^2 + 2a_-' a_+' \times \cos(2(\omega t - nkz) + \alpha_+' - \alpha_-')), \quad (109a)$$

$$\frac{n^2}{c^2}\mathbf{S}' = \epsilon_0 \frac{n^3}{c}(a_-'^2 + a_+'^2 + 2a_-' a_+' \times \cos(2(\omega t - nkz) + \alpha_+' - \alpha_-')). \quad (109b)$$

That is, both energy density U' and the momentum density $n^2\mathbf{S}'/c^2$ are fluctuating in space and time at twice the frequency of the incident electromagnetic wave. Note that the Poyning vector $\mathbf{S}'$ is proportional to the refractive index n:

$$\mathbf{S}' = \epsilon_0 n (a_-'^2 + a_+'^2 + 2a_-' a_+' \times \cos(2(\omega t - nkz) + \alpha_+' - \alpha_-')). \quad (110)$$

Averaging Eq. (108) over the optical period $\tau = 2\pi/\omega$,

$$\langle U' + \frac{n}{c}\mathbf{S}'\rangle_\tau = (1 + \mathbf{e}_3)\epsilon_0 n^2 (a_-'^2 + a_+'^2), \quad (111)$$

and separating the energy and momentum components, we arrive at

$$\langle U'\rangle_\tau = \epsilon_0 n^2 (a_-'^2 + a_+'^2), \quad (112a)$$

$$\frac{n^2}{c^2}\langle \mathbf{S}'\rangle_\tau = \mathbf{e}_3 \epsilon_0 \frac{n^3}{c}(a_-'^2 + a_+'^2). \quad (112b)$$

Thus, the average energy is proportional to the square of the refractive index n while the average momentum is proportional to the cube. In particular, the average Poynting vector is proportional to the refractive index n:

$$\langle \mathbf{S}'\rangle_\tau = \mathbf{e}_3 \epsilon_0 n c (a_-'^2 + a_+'^2). \quad (113)$$

4.4.2 Case 2: $n^2 < 0$

If $n^2 < 0$, we use the electromagnetic wave expression in Eq. (86). Its corresponding energy-momentum equation is

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}_\pm' \hat{E}_\pm'^\dagger = -\frac{1}{2}\epsilon_0 n^2 \hat{\mathbf{e}}_+ (\hat{a}_-' e^{-i\mathbf{e}_3\omega t} e^{\mp|n|kz} + \hat{a}_+' e^{i\mathbf{e}_3\omega t} e^{\pm|n|kz}) \hat{\mathbf{e}}_+^\dagger (\hat{a}_-' e^{-i\mathbf{e}_3\omega t} e^{\mp|n|kz} + \hat{a}_+' e^{i\mathbf{e}_3\omega t} e^{\pm|n|kz}). \quad (114)$$

Using the identities in Eqs. (104a), (104b), and (106), Eq. (114) becomes

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}_\pm' \hat{E}_\pm'^\dagger = (1 + \mathbf{e}_3)\epsilon_0 n^2 (\hat{a}_-'^* e^{i\mathbf{e}_3\omega t} e^{\mp|n|kz} + \hat{a}_+'^* e^{-i\mathbf{e}_3\omega t} e^{\pm|n|kz}) (\hat{a}_-' e^{-i\mathbf{e}_3\omega t} e^{\mp|n|kz} + \hat{a}_+' e^{-i\mathbf{e}_3\omega t} e^{\pm|n|kz}). \quad (115)$$

Distributing the terms in Eq. (115) yields

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}_\pm' \hat{E}_\pm'^\dagger = (1 + \mathbf{e}_3)\epsilon_0 n^2 (a_-'^2 e^{\mp 2|n|kz} + a_+'^2 e^{\pm 2|n|kz} \hat{a}_+'^* \hat{a}_-' e^{-2i\mathbf{e}_3\omega t} + \hat{a}_-'^* \hat{a}_+ e^{2i\mathbf{e}_3\omega t}). \quad (116)$$

Equation (116) is the expansion of the right side of Eq. (114).

To expand the left side of Eq. (114), we use the definition of the electromagnetic field $\hat{E}'$ for $n^2 < 0$ in Eq. (86), together with the :

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}_\pm' \hat{E}_\pm'^\dagger = -\frac{1}{2}\epsilon n^2 \left(\mathbf{E}_\pm' \pm \frac{1}{|n|}\zeta_0 \mathbf{e}_3 \mathbf{H}_\pm'\right) \left(-\mathbf{E}_\pm' \pm \frac{1}{|n|}\zeta_0 \mathbf{e}_3 \mathbf{H}_\pm'\right), \quad (117)$$

since

$$(\mathbf{e}_3 \mathbf{H}_\pm')^\dagger = \mathbf{e}_3^\dagger \mathbf{H}_\pm'^\dagger = \mathbf{e}_3 \mathbf{H}_\pm' \quad (118)$$

and the spatial inverse of a vector is its negative.

Distributing the terms in Eq. (117),

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}_\pm' \hat{E}_\pm'^\dagger = -\frac{1}{2}\epsilon_0 n^2 (-\mathbf{E}_\pm'^2 + \frac{\zeta_0^2}{|n|^2}\mathbf{e}_3 \mathbf{H}_\pm' \mathbf{e}_3 \mathbf{H}_\pm' \pm \frac{\zeta_0}{|n|}(\mathbf{E}_\pm' \mathbf{e}_3 \mathbf{H}_\pm' - \mathbf{e}_3 \mathbf{H}_\pm' \mathbf{E}_\pm')), \quad (119)$$

and using the perpendicularity assumption in Eq. (88), we get

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}_\pm' \hat{E}_\pm'^\dagger = \frac{1}{2}\epsilon_0 n^2 (\mathbf{E}_\pm'^2 + \frac{\zeta_0^2}{|n|^2}\mathbf{H}_\pm'^2 \pm \frac{\zeta_0}{|n|}\mathbf{e}_3 (\mathbf{E}_\pm' \mathbf{H}_\pm' + \mathbf{H}_\pm' \mathbf{E}_\pm')). \quad (120)$$

From Eqs. (89a) and (89b), we see that

$$\mathbf{E}'_{\pm} = \pm\frac{\zeta_0}{n}\mathbf{H}_{\pm}. \tag{121}$$

Using this equivalence in Eq. (120), we obtain

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}'_{\pm}\hat{E}'^{\dagger}_{\pm} = \epsilon_0 n^2(1+\mathbf{e}_3)\mathbf{E}'^2_{\pm}, \tag{122}$$

which is a simple expression in terms of $\mathbf{E}'^2_{\pm}$. Comparing Eq. (122) with Eqs. (99a) and (99b) leads to the relation

$$-\frac{1}{2}\epsilon_0 n^2 \hat{E}'_{\pm}\hat{E}'^{\dagger}_{\pm} = U_{\pm} + \mathbf{s}'_{\pm}, \tag{123}$$

where

$$\mathbf{s}'_{\pm} = \epsilon_0 n^2 \mathbf{e}_3 \mathbf{E}'^2_{\pm} \tag{124}$$

is proportional to momentum density. Thus, even if the wave's Poynting vector is zero,

$$\mathbf{S}'_{\pm} = \mathbf{E}_{\pm} \times \mathbf{H}'_{\pm} = 0, \tag{125}$$

because $\mathbf{E}'_{\pm}$ and $\mathbf{H}'_{\pm}$ are parallel by Eq. (121) as noted by Budden, the electromagnetic wave still possess a quantity $\mathbf{s}'_{\pm}$ with units of energy, which we interpret as proportional to momentum density . This is a new result.

Equating Eqs. (123) with (116), we get

$$U'_{\pm} + \mathbf{s}'_{\pm} = (1+\mathbf{e}_3)\epsilon_0 n^2 (a'^2_{-} e^{\mp 2|n|kz} + a'^2_{+} e^{\pm 2|n|kz} \\ \hat{a}'^{*}_{+}\hat{a}'_{-} e^{-i\mathbf{e}_3 2\omega t} + \hat{a}'^{*}_{-}\hat{a}_{+} e^{i\mathbf{e}_3 2\omega t}). \tag{126}$$

Expanding the exponentials,

$$U'_{\pm} + \mathbf{s}'_{\pm} = (1+\mathbf{e}_3)\epsilon_0 n^2 (a'^2_{-} e^{\mp 2|n|kz} + a'^2_{+} e^{\pm 2|n|kz} \\ + 2a'_{+}a'_{-}\cos(2\omega t + \alpha'_{+} - \alpha'_{-})), \tag{127}$$

and separating the scalar and vector parts, we arrive at

$$U'_{\pm} = \epsilon_0 n^2 (a'^2_{-} e^{\mp 2|n|kz} + a'^2_{+} e^{\pm 2|n|kz} \\ 2a_{+}a_{-}\cos(2\omega t + \alpha'_{+} - \alpha_{-}), \tag{128a}$$

$$\mathbf{s}_{\pm} = \mathbf{e}_3 \epsilon_0 n^2 (a'^2_{-} e^{\mp 2|n|kz} + a'^2_{+} e^{\pm 2|n|kz} \\ 2a_{+}a_{-}\cos(2\omega t + \alpha'_{+} - \alpha_{-}). \tag{128b}$$

Equations (128a) and (128b) are the desired relations for energy and momentum densities for the case $n^2 < 0$. ince $n^2 < 0$, then both the energy density $U'_{\pm}$ would be negative, and $\mathbf{s}_{\pm}$ (a quantity we claim to be proportional to momentum density) would be pointing in the direction opposite to $\mathbf{e}_3$, the direction of motion and momentum of the wave if $n^2 > 0$.

Taking the time average of the energy and momentum relations in Eqs. (128a) and (128b) over the period $\tau = 2\pi/\omega$, we obtain

$$U'_{\pm} = \epsilon_0 n^2 (a'^2_{-} e^{\mp 2|n|kz} + a'^2_{+} e^{\pm 2|n|kz}), \tag{129a}$$

$$\mathbf{s}'_{\pm} = \epsilon_0 n^2 (a'^2_{-} e^{\mp 2|n|kz} + a'^2_{+} e^{\pm 2|n|kz}). \tag{129b}$$

Depending on the choice of the sign of the refractive index $n = \pm i\mathbf{e}_3|n|$, either a'_{-} or a'_{+} will dominate as $z >> 1/|n|k$.

4.4.3 Case 3: $n^2 = 0$

For $n^2 \to 0^{+}$, the electromagnetic wave $\hat{E}'$ is given in Eq. (94). To find its energy-momentum, we set $n = 0$ in Eq. (108) to obtain

$$U' = \frac{n^2}{c^2}\mathbf{S}' = 0. \tag{130}$$

On the other hand for $n^2 \to 0^{-}$, the electromagnetic wave $\hat{E}'_{\pm}$ is given in (86). To find its energy-momentum, we set $n = 0$ in Eq. (126) to get

$$U'_{\pm} = \mathbf{s}'_{\pm} = 0. \tag{131}$$

Regardless of the direction of evaluation of the limit of $|n| \to 0$, we get the same conclusion: the electromagnetic wave possesses neither energy nor momentum.

5 Conclusions

In this paper we used geometric algebra for computing the energy and momentum densities of an elliptically polarized wave in plasma for different cases of refractive indices: $n^2 > 0$, $n^2 < 0$, and $n^2 = 0$.

In Section 2, we began by introducing Clifford (geometric) algebra $\mathcal{C}l_{3,0}$ and showed that the direct product of two vectors can be expressed as a sum of their dot product and their imaginary cross product,

with the unit imaginary number as the unit trivector, which is a product of three basis vectors along the $x-$, $y-$, and $z-$directions. We showed that the exponential of an imaginary vector may be used as a vector rotation operator, for vector operands that are perpendicular to the vector argument of the exponential.

In Section 3, we summarized the fundamental equations of Electrodynamics. The first equation is the Maxwell's equation, which states that the spacetime derivative of the electromagnetic field is proportional to the charge-current density. The second equation is the wave equation, which is obtained by taking the derivative of the Maxwell's equation with respect to the spatial inverse of the spacetime derivative operator. Here the current density is a sum of the electric field vector and the imaginary magnetic field vector. We showed that the product of the electromagnetic field with its spatial inverse is proportional to the energy-momentum of the electromagnetic field.

In Section 4, we deduced the orbital motion of a charge subject to the electric field of a circularly polarized electromagnetic wave. We showed that the motion of the charge is a superposition of a linear motion and an elliptical motion, with the latter opposite to the forcing electric field. We took the time derivative of the charge's position vector to obtain the current density. We showed that the current density has two components: a constant current density and a time-varying current density proportional to the time-derivative of the electric field of the electromagnetic wave. We dropped the constant current density term, because we felt that this is only an artifact of our not also considering the magnetic field of the electromagnetic wave in the Lorentz force on the charge. We also assumed that the charge density is nearly zero. In this way, we showed that we can define new electric and magnetic fields in plasma which obeys a homogeneous wave equation and a homogeneous Maxwell's equation, with refractive indices computed from the proportionality of the time derivative of the current density to the second time derivative of the forcing electric field of the electromagnetic wave.

We assumed an electromagnetic wave similar in form to that in vacuum for different cases of the refractive index n. For $n^2 > 0$, we showed that the electric and magnetic fields are perpendicular; while for $n^2 < 0$, the two fields are parallel. For $n^2 > 0$, the energy-momentum of the wave is well-defined, which is a sum of the energy density and the Poynting vector. For $n^2 < 0$, the energy density is easy to interpret, since it is just proportional to the square of the wave's electric field. The momentum density is more difficult to interpret since the Poynting vector is zero, because the electric and magnetic fields are parallel, as noted by Budden. Nevertheless, we showed that we can construct a new quantity with units of energy that points along the direction of propagation or rotation of the wave; we claim that this is proportional to momentum density. But since the energy density and the new momentum density are both proportional to $n^2 < 0$, then the energy density is negative and the new momentum density is opposite to the direction of propagation of the wave for the case $n^2 > 0$.

There are many results in the paper which may not be physically correct, such as drifting charges under polarized light, negative energy densities of electromagnetic waves, and the exponential increase of wave amplitude as function of propagation distance. We hope to remove all these problems in our succeeding paper where we compute the refractive index from the the orbit of charges under the action of both the electric and magnetic field components of the elliptically polarized electromagnetic waves. We shall also try to extend our geometric algebra formalism to the reformulation of Appleton's magnetoionic theory of the ionosphere and the meaning of the complex refractive index.

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